LIMPOPO

PROVINCIAL GOVERNMENT

REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF

EDUCATION

CAPRICORN NORTH DISTRICT

GRADE 12

MATHEMATICS P2

DATE OF ISSUE: 12 MARCH 2024

LEVEL 1, 2, 3 & 4 CONTENT QUESTIONS

DEPARTMENT OF EDUCATION

CAPRICORN NORTH DISTRICT

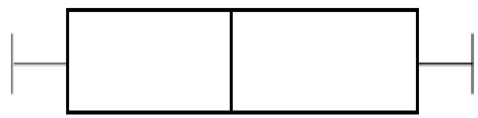
MATHEMATICS

STATISTICS

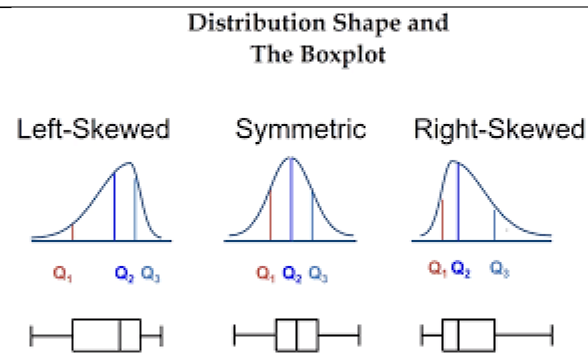
PAPER 2

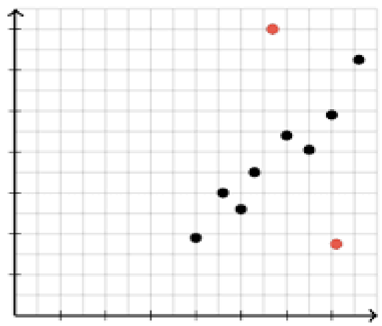
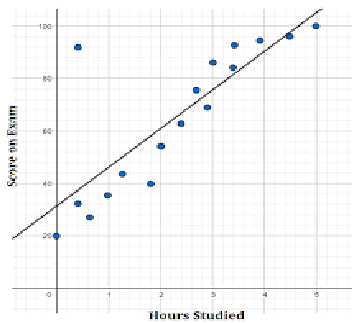
CHAPTER 0NE

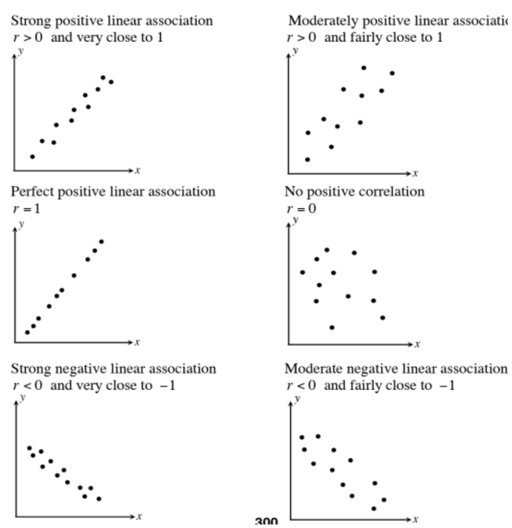
	HOW TO LEARN IT?	RELEVANT FORMULAE AND KEYWORDS	
Measures of central tendency	: the most common value in the data set, thevalue that appears most. Median ($\mathbf{Q_2}$): the middle value in an ordered data set. Mean ($\bar{x}$S): the average of the data set.	UNGROUPED DATA $\bar{x} = \frac{sum\ of\ values\ in\ the\ data\ set}{number\ of\ values\ in\ the\ data\ set}$ $x = \frac{\sum x}{n}$	GROUPED DATA $\bar{x} = \frac{\sum f.x}{\sum f}$ Modal class: class or interval with highest frequency Median class: $\frac{n+1}{2}$ (Represent the median class position)
Measures of dispersion or spread	Range: the difference between the largest value andthe smallest value in a data set. Interquartile range (IQR): the difference betweenthe upper quartile and the lower quartile.	$Range = largest\ value - smallest\ value$ $IQR = Q_3 - Q_1$	
Five – number summary	Minimum: the smallest value in the data set. Lower quartile ($\boldsymbol{Q_1}$): the medina of the lower half ofthe ordered data. Median ($\boldsymbol{Q_2}$): the value that divides the data intohalves Lower quartile ($\boldsymbol{Q_3}$): the medina of the upper half ofthe ordered data. Maximum: the largest value in the data set.		Position of Lower quartile $Q_1 = \frac{n+1}{4}$ Upper quartile $Q_3 = \frac{3(n+1)}{4}$ Calculating mean using calculator:

	Order your data in ascending order when data isungrouped to find the five-number summary.	Minimum value Lower quartile (Q_1) Median (Q_2) Upper Quartile(Q_3) Maximum value	SHIFT Select: MODE Scroll down Select :3 Select: 1 Press: Mode Select 1(enter data) Press: AC Press: Shift 1 Press :4
Box and whisker diagram	Plot the five – number summary to represent the boxand whisker plot. Remember this is a very accurate (on a scale)representation of your data.	Lowest value, Q_1, Q_2, Q_3 and Highest value 	
	We can read the following from the box and whiskerplot: A measure of central tendency (median). Minimum and maximum, lower quartile andupper quartile. The spread of the middle half of the data (thewidth of the box or IQR) The range (the width of the whiskers). The skewness.		
Cumulative frequency	Determining cumulative frequency is an effective way of representing grouped data.		
Percentiles	A percentile is the value which a specific percentage of data elements lie.		
	Plot the correct coordinates.	Remember to plot accurate and to ground yourgraph.	

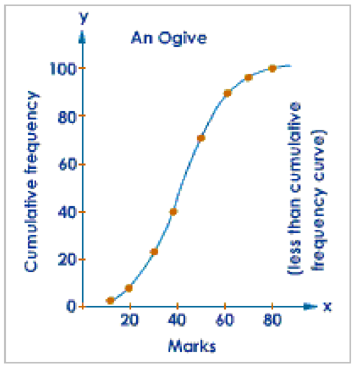
	Remember to ground the ogive. Be able to read the quartiles and percentiles from theogive.		
	Practice by drawing the ogive with the aboveinformation. Remember to use graph paper because it is an exactdrawing!	Draw the ogive with info from the cumulative frequency table.	
Variance and standard deviation	Variance = σ^2 Standard deviation = σ	Use your calculator to determine these values.	
	You must be able to work out data that fall within or outside and upto three standard deviations from the mean. One standard deviation: ($\bar{x} - 1\sigma$s ; $\bar{x} + 1\sigma$) Two standard deviation: ($\bar{x} - 2\sigma$; $\bar{x} + 2\sigma$) Three standard deviation: ($\bar{x} - 3\sigma$; $\bar{x} + 3\sigma$)	How to calculate on the calculator. Use a calculator for a Casio fx – 82ZA PLUS.Press [MODE] Select [2: STAT] Select 1 :Put in	

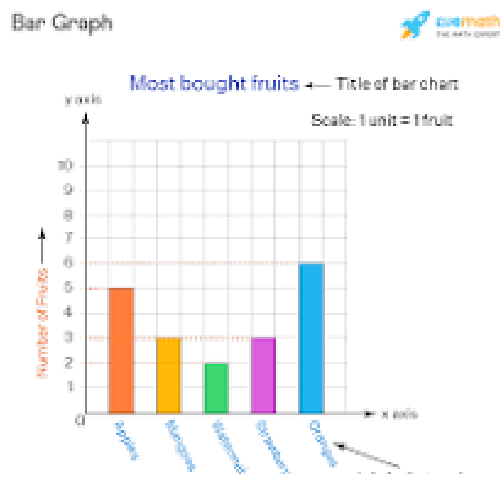
		data Press AC SHIFF ;1 Press 4 1: n 2: $\bar{X}$ 4: VAR 3: σx	
Symmetrical data and skewed data	A symmetrical data set is balanced, or nearly so, oneither side of the median. Skewed data is spread out more on one side of the median than on the other side. If mean ≈ $\boldsymbol{Q_2}$: the data is symmetrical. If mean > $\boldsymbol{Q_2}$: the distribution is skewed to the right, or positively skewed. If mean < $\boldsymbol{Q_2}$: the distribution is skewed to the left, or negatively skewed.		

Scatter plot	A scatter plot is a plot of bivariate data (data that hastwo variables) which shows a relationship between the two sets of data.	$\hat{y}= a + bx$	
Outlier	An outlier is a data value which is much larger orsmaller than the rest of the values in the data set. 	A value is defined to be an outlier. The value is less than $Q1- 3/\ 2\ IQR$ The value is greater than $Q3+ 3\ /2\ IQ$R	
Least squares regression line	A regression line (line of best fit) is used to show the general trend which a set of data follows. $\hat{y}= a + bx$ 	Use a calculator for a Casio fx – 82ZA PLUS.Press [MODE] Select: :[STAT] Select [2: A + Bx] Put in x and y valuesPress AC [SHIFT 1] (STAT)	

		5: Reg 1: A 2: B	
	Learn to calculate ‘a’ ‘b’ and substitute in the regression equation.	Never join the points on a scatter plot; use a line ofbest fit. Make sure you know how to draw in the line.Make use of a mean point($\overline{x}$; $\overline{y}$) Plot the mean point. Joint the mean point and the y – intercept from $\hat{y}= a + bx$	
Correlation co-efficient	Correlation co-efficient is represented with ‘r’.This is a value between -1 and 0 or 0 and 1.  	Use a calculator for a Casio fx – 82ZA PLUS.Press [MODE] Select [2:STAT] Select [2: A + Bx] Put in x and y values. Press AC [SHIFT 1] (STAT) 5: Reg 3: r	

<table>
<tr>
<td>GRAPHS</td>
<td>OGIVE: is a graph that shows the information in a cumulative frequency table. The graph takes the S shape. The graph is useful for estimating the median and inter quartile range of the grouped data.


</td>
<td>Bar graph: is a representation of ungrouped data that does not have to be numerical. There is generally a gap between the bars.


</td>
<td>Histograms: is a graphs that represent grouped data.
There is no gap between the bars.

Frequency polygon: can be used instead of a histogram for illustrating grouped data

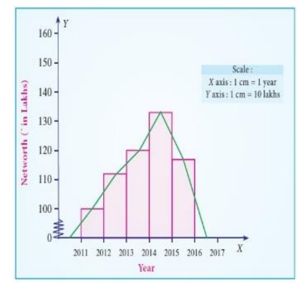


Fig 4.10 Frequency polygon for Net Worth in the years 2011-2016</td>
</tr>
</table>

COMMON ERRORS AND MISCONCEPTIONS

- After plotting points on scatterplot, learners usually join the points
- Learners fails to identify the outlier on scatterplot
- Learners cannot identify the dependent and independent variables
- Learners are unable to calculate the unknown variables of data when given mean
- Some learners do not round off their answers to two decimal place
- Learners are not careful when capturing data in the calculators which lead to incorrect answers for mean and standard deviation.
- Learners cannot interpret the skewedness of box and whisker diagram
- Learners don't ground the ogive graphs
- Some learners use ruler to sketch the ogive graph instead of free hand

STATISTICS EXAM GUIDELINE

1. Candidates should be encouraged to use the calculator to calculate standard deviation, variance and the equation of the least squares regression line.

2. The interpretation of standard deviation in terms of normal distribution is not examinable.

3. Candidates are expected to identify outliers intuitively in both the scatter plot as well as the box and whisker diagram.
   In the case of the box and whisker diagram, observations that lie outside the interval (lower quartile – 1,5 IQR; upper quartile + 1,5 IQR) are considered to be outliers. However, candidates will not be penalized if they did not make use of this formula in identifying outliers.

LEVEL 1 AND 2 QUESTIONS

QUESTION 1

During the month of July, a number of parents visited a local clinic suffering from influenza. The table below shows the cumulative number of patients treated as per the dates given.

Dates in the month of July	3	5	8	12	15	19	22	26
Number of patients treated	270	275	376	420	602	684	800	820

1.1	Draw a scatter plot of the above data.	(3)
1.2	Determine the equation of the least squares line for the data.	(4)
1.3	Draw the least squares line for the data.	(2)
1.4	Estimate how many patients were as on 30 June.	(2)
1.5	Estimate how many patients were as on 24 July	(2)
1.6	Determine the correlation coefficient for the data. Interpret this result	(3)

[16]

QUESTION 2

A learner conducted an experiment to investigate the relationship between age and resting heart rate (in beats per minute). He sought the assistance of the local clinic. The information for 12 people is shown in the table below.

Age (years)	59	32	42	50	22	39	21	20	27	40	29	47
Resting heart rate (beats per minute)	88	74	74	93	85	71	78	82	70	75	95	75

2.1	Represent the data in a scatter plot.	(3)
2.2	Determine the equation of the least squares line.	(4)
2.3	Draw the least squares line on the scatter plot.	(2)
2.4	Calculate the correlation coefficient for the data.	(2)
2.5	Use the correlation coefficient to comment on the relationship between age and the resting heart rate.	(2)
2.6	If a learner uses the least squares line to predict the resting heart rate of a 45-year-old person, will his answer be reliable? Motivate your answer.	(2)

[15]

Q

QUESTION 3

The term *latitude* refers to how far a place is from the equator. Latitude in the Northern Hemisphere ranges from 0° at the equator to 90° N at the north pole.
Below are the latitudes of several cities in the Northern Hemisphere together with the mean maximum temperature for April in degrees Celsius.

City	Northern Latitude	Mean maximum temperature for April
Lagos, Nigeria	6	32
London, England	52	13
Clacutta, India	23	36
Rome, Italy	42	20
Moscow, Russia	56	8
Cairo, Egypt	30	28
San Juan, Puerto Rico	18	29
Copenhagen, Denmark	56	10
Tokyo, Japan	35	17

3.1	Draw a scatter plot for the above information on DIAGRAM SHEET attached.	(3)
3.2	Determine the equation of the least squares regression line.	(4)
3.3	Draw the least squares regression line on your scatter plot, on DIAGRAM SHEET attached.	(2)
3.4	What information does the y-intercept of this line represent?	(1)
3.5	The city of Madrid has a latitude of 40°N. Determine the mean maximum temperature for April for this city'.	(2)
3.6	Calculate the correlation coefficient of the data.	(2)
3.7	Explain the correlation between latitude and the mean maximum temperature for April.	(1)

[15]

Q

QUESTION 4

The truck shop at Great Future School sells cans of soft drinks. The Environment Club at the school decided to have a can-collection project for three weeks to make learners aware of the effects of litter on the environment.

The data below shows a number of cans collected on each school day of the three-week project.

58 83 85 89 94 97 98 100 105 109 112 113 114 120 145

4.1	Calculate the mean number of cans collected over the three-week period.	(2)
4.2	Calculate the standard deviation.	(2)
4.3	Determine the lower and upper quartiles of the data.	(2)
4.4	Use the scale line on DIAGRAM SHEET to draw a box and whisker diagram to represent the data.	(3)

[9]

QUESTION 5

A large company employs 9 salespersons. The commission that each salesperson earned (in rands) in a certain month is shown below:

3 900 5 700 7 300 10 600 13 000 13 600 15 100 15 800

5.1	Calculate the mean of the above data	(2)
5.2	Calculate the standard deviation for the data.	(2)

QUESTION 6

The table below gives a breakdown of the PSL log standings for the 8 top teams at the end of 2008/2009.

POSITION	TEAM	POINTS
1	Super Sport United	55
2	Orlando Pirates	55
3	Kaizer Chiefs	50
4	Free State Stars	47
5	Golden Arrows	x
6	Bidvest Wits	x
7	Ajax Cape Town	x
8	Amazulu Royals	42

6.1 If the average points for these 8 teams is 48,375, show that $x = 46$ (2)

6.2 Draw a box and whisker diagram of the information given on DIAGRAM SHEET. (2)

QUESTION 7

The data below shows the total monthly rainfall (in millimetres) at Cape Town International Airport for the year 2002.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
60,9	14,9	9,3	28,0	71,9	76,4	98,2	65,7	26,1	32,5	23,6	15,0

7.1 Determine the mean monthly rainfall for 2002. (2)

7.2 Write down the five-number summary for the data (5)

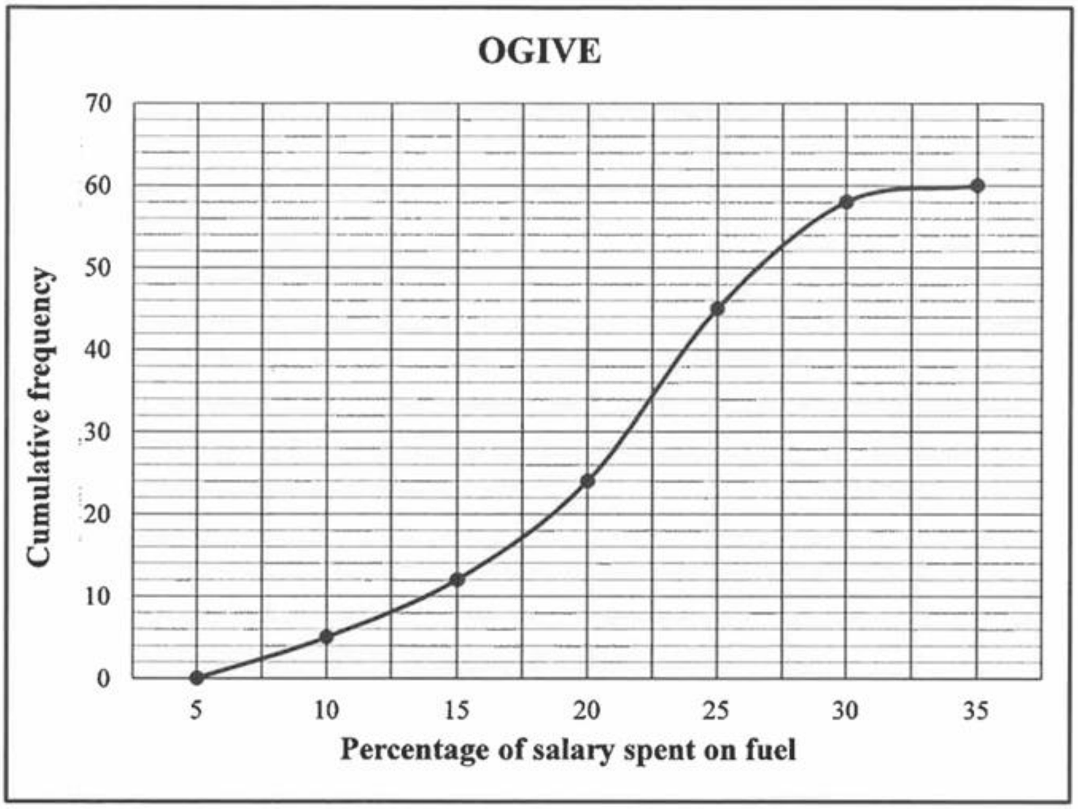
7.4 Draw a box and whisker diagram for the data on DIAGRAM SHEET. (3)

7.4 By making reference to the box and whisker diagram, comment on the spread of the rainfall for the year. (2)

7.5 Calculate the standard deviation for the data. (3)

QUESTION 8

A company conducted research among all its employees on what percentage of their monthly salary was spent on fuel in a particular month. The data is represented in the ogive (cumulative frequency graph) below.



8.1 How many people are employed at this company? (1)

8.2 Write down the modal class of the data. (1)

8.3 How many employees spent more than 22,5% of their monthly salary on fuel? (2)

8.4 An employee spent R2 400 of his salary on fuel in that particular month. (2)

Determine the monthly salary of this employee if he spends 7% of his salary on fuel.

QUESTION 9

A survey was conducted among 100 people about the amount that they paid on a monthly basis for their cell phone contracts. The person carrying out the survey calculated the estimated mean to be R309 per month. Unfortunately, he lost some of the data thereafter. The partial results of the survey are shown in the frequency table below:

AMOUNT PAID (IN RANDS	FREQUENCY
o < x < 100	7
100 < x < 200	12
200 < x < 300	A
300 < x < 400	35
400 < x < 500	B
500 < x < 600	6

9.1 How many people paid R200 or less on their monthly cellphone contracts? (1)

9.2 Use the information above to show that a = 24 and b = 16. **(L3)** (5)

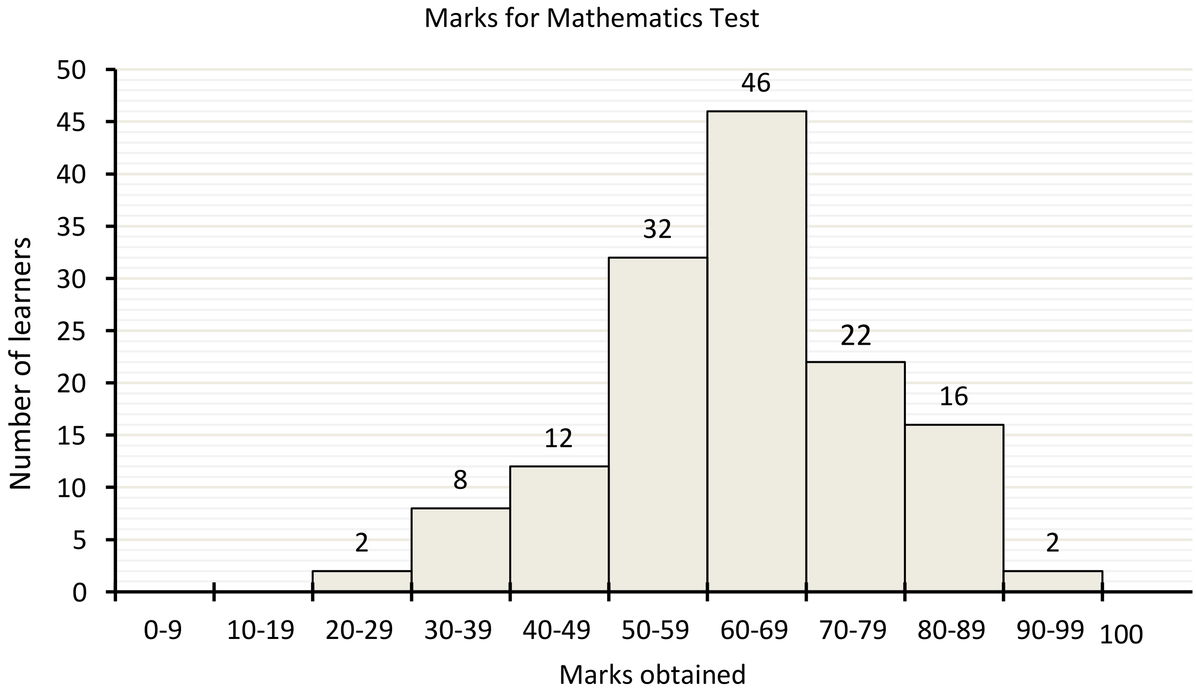
9.3 Write down the modal class for the data. (2)

9.4 On the grid provided in the ANSWER BOOK, draw an ogive (cumulative frequency graph) to represent the data (2)

9.5 Determine how many people paid more than R420 per month for their cellphone contracts. (2)

QUESTION 10

The marks obtained by learners of a certain school in a Mathematics test is represented in the histogram below:



10.1	How many learners wrote the test?	(1)
10.2	Write down the modal class.	(1)
10.3	Draw the ogive for the given information	(4)
10.4	Use the ogive to estimate the interquartile range	(2)

QUESTION 11

A bakery kept a record of the number of loaves of bread a tuck-shop ordered daily over the last 18 days. The information is shown in the table below.

10	11	13	14	14	15	16	18	18
19	19	20	21	35	35	37	40	41

11.1 Calculate the:

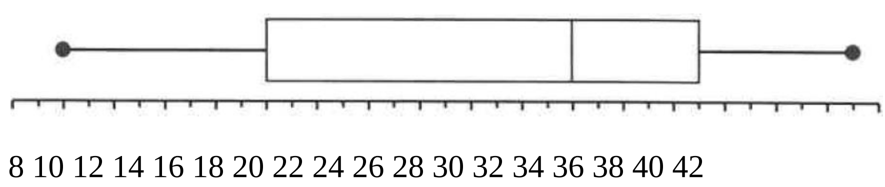
11.1.1 Mean number of loaves of bread ordered daily (2)

11.1.2 Standard deviation of the data (1)

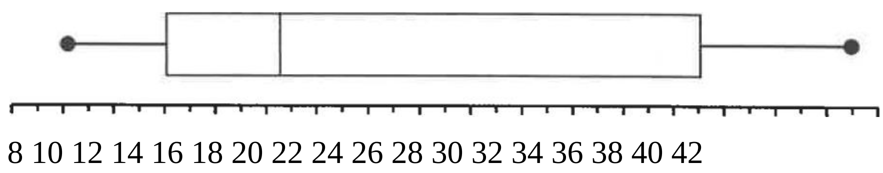
11.2 The tuck-shop owner was not able to sell all the loaves of bread delivered daily. He calculated the mean number of loaves sold over the 18 days to be 20. Calculate the number of loaves of bread which were NOT sold over the 18 days. (2)

11.3 One of the two box and whisker diagrams drawn below represents the data given in the table above.

Graph A:



Graph B:



11.3.1 Which ONE of the two box and whisker diagrams, drawn above, correctly represents the data? Write down a reason for your answer. (2)

11.3.2 Describe the skewness of the data. (1)

QUESTION 12

The table below shows the weight (to the nearest kilogram) of each of the 27 participants in a weight-loss programme.

56	68	69	71	71	72	82	84	85
88	89	90	92	93	94	96	97	99
102	103	127	128	134	135	137	144	156

12.1 Calculate the range of the data. (2)

12.2 Write down the mode of the data. (1)

12.3 Determine the median of the data. (1)

12.4 Determine the interquartile range of the data. (3)

12.5 Use the number line provided in the ANSWER BOOK to draw a box and whisker diagram for the data above. (2)

12.6 Determine the standard deviation of the data. (2)

QUESTION 13

The following data shows the ages of 10 people who donated blood in December 2012.

25 47 40 34 28 x 37 28 55 30

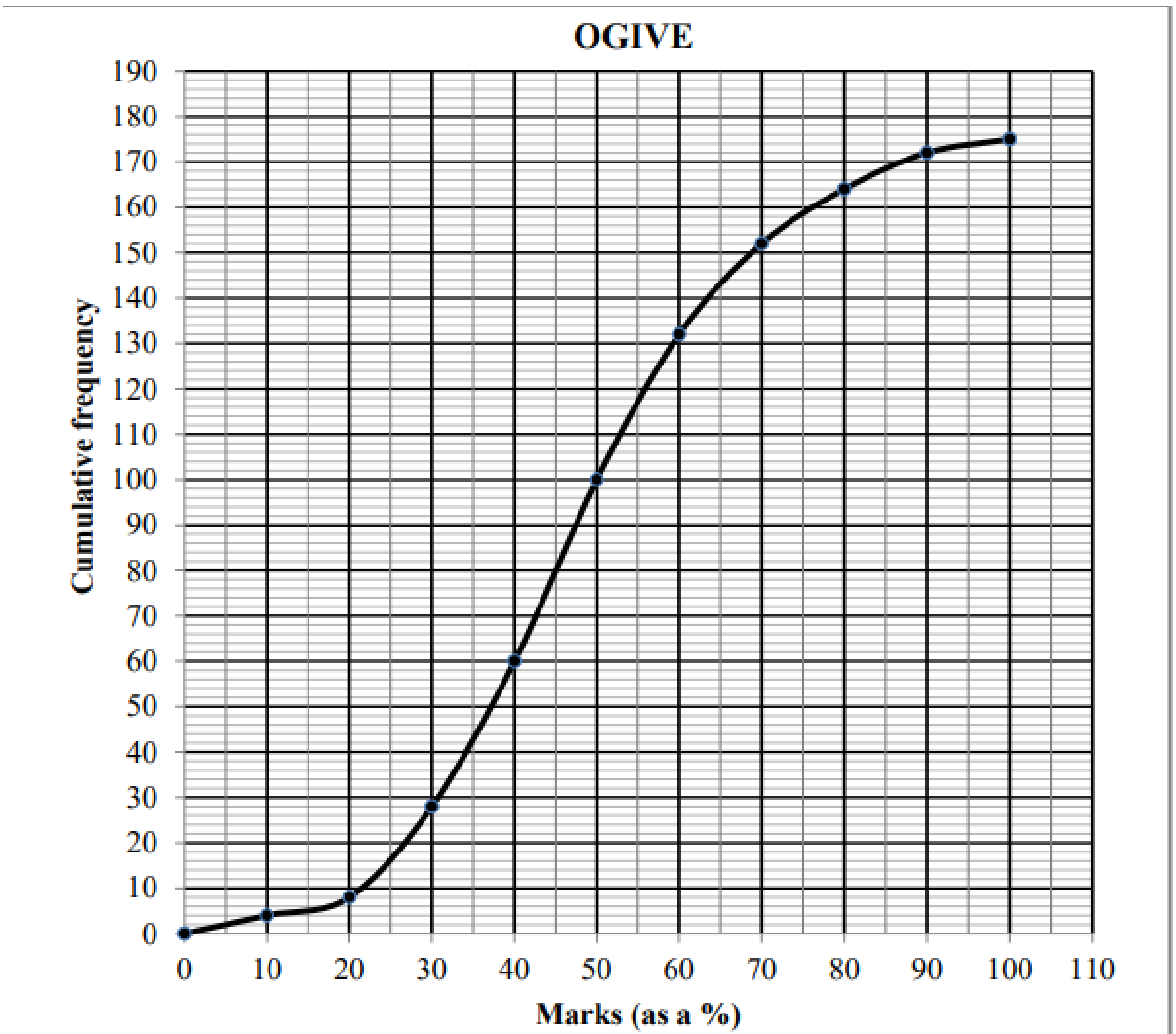
13.1 Determine the mean in terms of x. (1)

13.2 Determine the value of x if the mean is 36. Show ALL calculations. (2)

13.3 Hence, determine the standard deviation. (2)

QUESTION 14

15 learners from various schools wrote an aptitude test in order to qualify for a bursary. Their marks (as a percentage) are represented in the ogive (cumulative frequency graph)



14.1.1 How many learners wrote the test? (1)

14.1.2 Write down the modal class of the data. (1)

14.1.3 The minimum mark to qualify for a bursary is 75%. How many learners qualified for a bursary? (2)

14.2 The table below shows the marks that 15 learners from one particular school obtained in the aptitude test

Marks (as a %)	62	58	78	85	74	48	74	84	100	46	80	92	60	90	92

Calculate the:

14.2.1 Mean mark obtained by these learners (2)

14.2.2 Standard deviation of these learners' marks (1)

[12]

THANK YOU

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CHAPTER 0NE

LEVEL 3 AND 4 QUESTIONS

The truck shop at Great Future School sells cans of soft drinks. The Environment Club at the school decided to have a can-collection project for three weeks to make learners aware of the effects of litter on the environment.

The data below shows a number of cans collected on each school day of the three-week project.

58 83 85 89 94 97 98 100 105 109 112 113 114 120 145

5.1 Calculate the mean number of cans collected over the three-week period. (2)

5.2 Calculate the standard deviation. (2)

5.3 Determine the lower and upper quartiles of the data. (2)

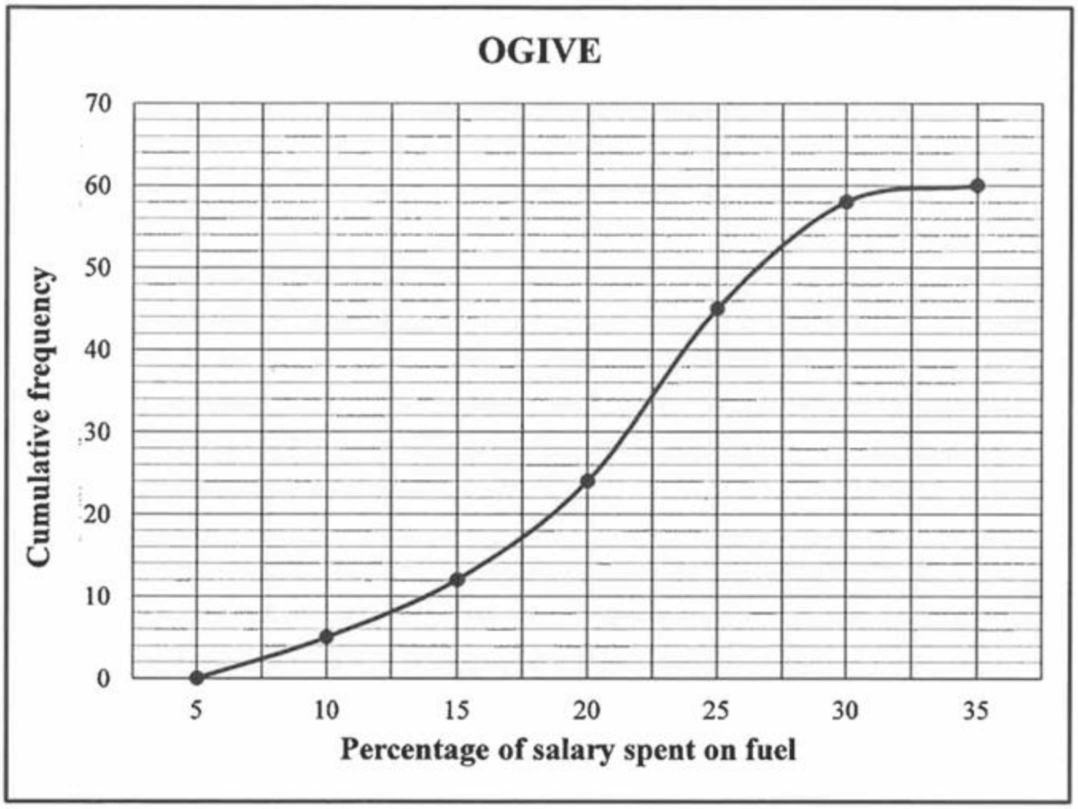
5.4 Use the scale line on DIAGRAM SHEET to draw a box and whisker diagram to represent the data. (3)

5.5 On how many days did the number of cans collected lie outside ONE standard deviation of the mean. (3)

[12]

QUESTION 1

A company conducted research among all its employees on what percentage of their monthly salary was spent on fuel in a particular month. The data is represented in the ogive (cumulative frequency graph) below.



1.1 How many people are employed at this company? (1)

1.2 Write down the modal class of the data. (1)

1.3 How many employees spent more than 22, 5% of their monthly salary on fuel? (2)

1.4 An employee spent R2 400 of his salary on fuel in that particular month. (2)

Determine the monthly salary of this employee if he spends 7% of his salary on fuel.

1.5 The monthly salaries of these employees remains constant and the number of litres of fuel used in each month also remains constant. If the fuel price increases from R21, 43 per litre to R22, 79 per litre at the beginning of the next month, how will the above ogive change? (2)

QUESTION 2

A survey was conducted among 100 people about the amount that they paid on a monthly basis for their cell phone contracts. The person carrying out the survey calculated the estimated mean to be R309 per month. Unfortunately, he lost some of the data thereafter. The partial results of the survey are shown in the frequency table below:

AMOUNT PAID (IN RANDS	FREQUENCY
o < x < 100	7
100 < x < 200	12
200 < x < 300	A
300 < x < 400	35
400 < x < 500	B
500 < x < 600	6

2.1 How many people paid R200 or less on their monthly cellphone contracts? (1)

2.2 Use the information above to show that a = 24 and b = 16. (5)

2.3 Write down the modal class for the data. (2)

2.4 On the grid provided in the ANSWER BOOK, draw an ogive (cumulative frequency graph) to represent the data (2)

2.5 Determine how many people paid more than R420 per month for their cellphone contracts. (2)

QUESTION 3

A bakery kept a record of the number of loaves of bread a tuck-shop ordered daily over the last 18 days. The information is shown in the table below.

10	11	13	14	14	15	16	18	18
19	19	20	21	35	35	37	40	41

3.1 Calculate the:

3.1.1 Mean number of loaves of bread ordered daily (2)

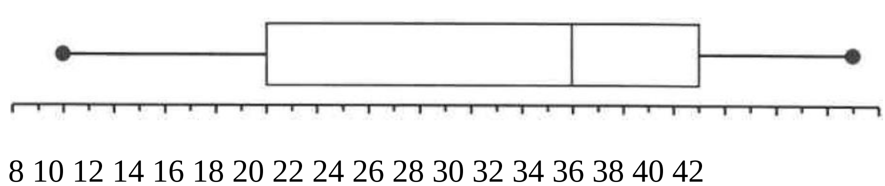
3.1.2 Standard deviation of the data (1)

3.1.3 Number of days on which the number of loaves of bread ordered was more than one standard deviation above the mean (2)

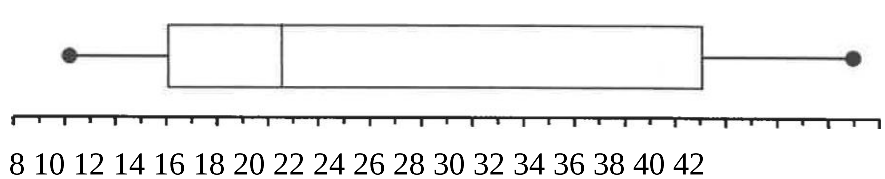
3.2 The tuck-shop owner was not able to sell all the loaves of bread delivered daily. He calculated the mean number of loaves sold over the 18 days to be 20. Calculate the number of loaves of bread which were NOT sold over the 18 days. (2)

3.3 One of the two box and whisker diagrams drawn below represents the data given in the table above.

Graph A:



Graph B:



3.3.1 Which ONE of the two box and whisker diagrams, drawn above, correctly represents the data? Write down a reason for your answer. (2)

3.3.2 Describe the skewness of the data. (1)

QUESTION 4

The table below shows the weight (to the nearest kilogram) of each of the 27 participants in a weight-loss programme.

56	68	69	71	71	72	82	84	85
88	89	90	92	93	94	96	97	99
102	103	127	128	134	135	137	144	156

4.1 Calculate the range of the data. (2)

4.2 Write down the mode of the data. (1)

4.3 Determine the median of the data. (1)

4.4 Determine the interquartile range of the data. (3)

4.5 Use the number line provided in the ANSWER BOOK to draw a box and whisker diagram for the data above. (2)

4.6 Determine the standard deviation of the data. (2)

4.7 The person weighing 127 kg states that she weighs more than one standard deviation above the mean. Do you agree with this person? Motivate your answer with calculations.(3)

QUESTION 5

The following data shows the ages of 10 people who donated blood in December 2012.

25 47 40 34 28 x 37 28 55 30

5.1 Determine the mean in terms of x. (1)

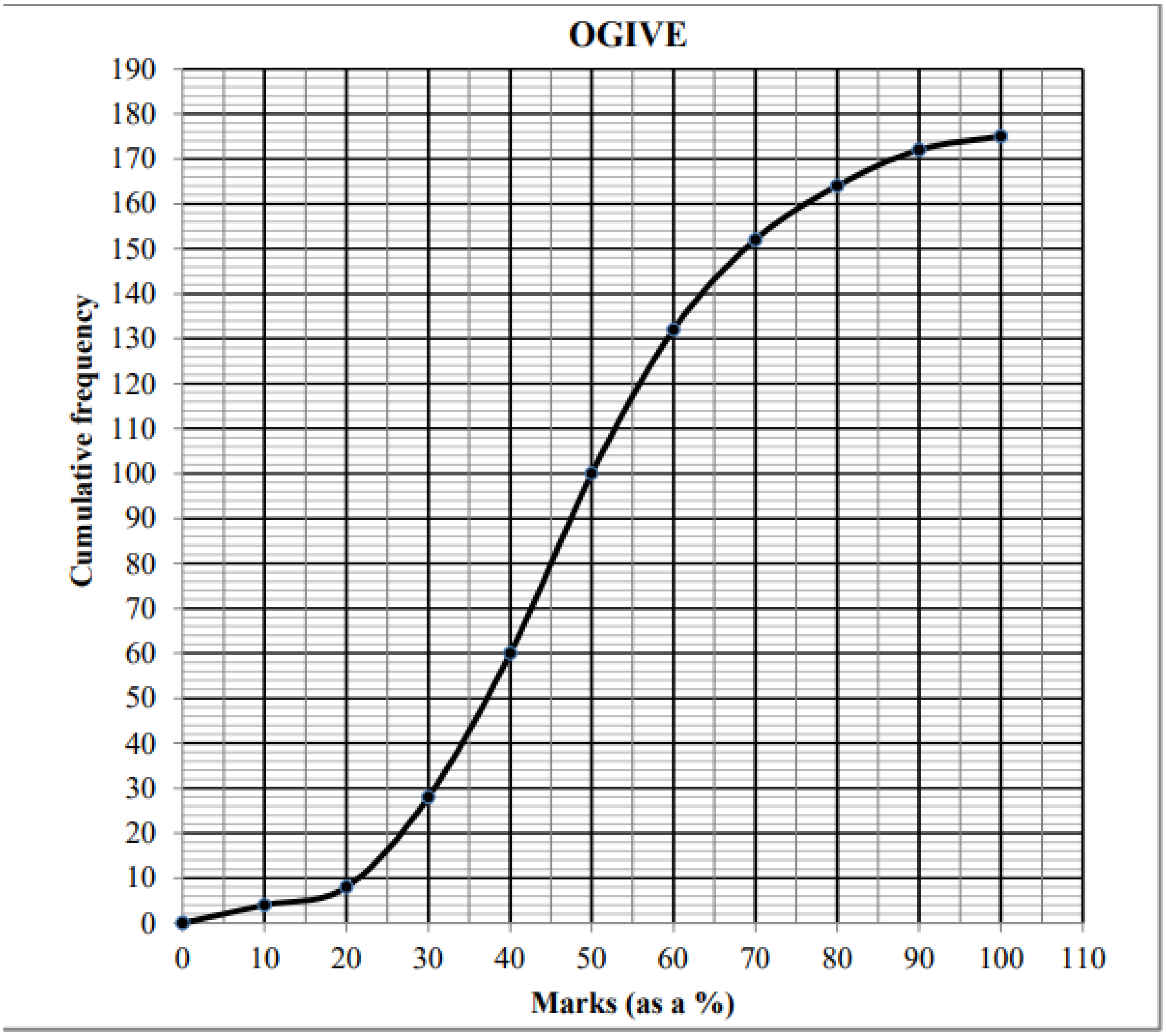
5.2 Determine the value of x if the mean is 36. Show ALL calculations. (2)

5.3 Hence, determine the standard deviation. (2)

5.4 How many people have ages which differ from the mean by more than one standard deviation? (2)

QUESTION 6

15 learners from various schools wrote an aptitude test in order to qualify for a bursary. Their marks (as a percentage) are represented in the ogive (cumulative frequency graph)



6.1.1 How many learners wrote the test? (1)

6.1.2 Write down the modal class of the data. (1)

6.1.3 The minimum mark to qualify for a bursary is 75%. How many learners qualified for a bursary? (2)

6.2 The table below shows the marks that 15 learners from one particular school obtained in the aptitude test

Marks (as a %)	62	58	78	85	74	48	74	84	100	46	80	92	60	90	92

Calculate the:

6.2.1 Mean mark obtained by these learners (2)

6.2.2 Standard deviation of these learners' marks (1)

6.2.3 Number of these learners whose marks lie more than one standard deviation above the mean

6.3 The final Grade 11 marks (as a percentage) obtained by a group of learners was analysed. The one standard deviation interval about the mean was calculated as (82,7; 94,1). Calculate the standard deviation for the final Grade 11 marks. (3)

[12]

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

CAPRICORN NORTH DISTRICT

ANALYTICAL GEOMETRY

MATHEMATICS GARDE 12

GRADE 12

ACTIVITIES MANUAL

ANALYTICAL

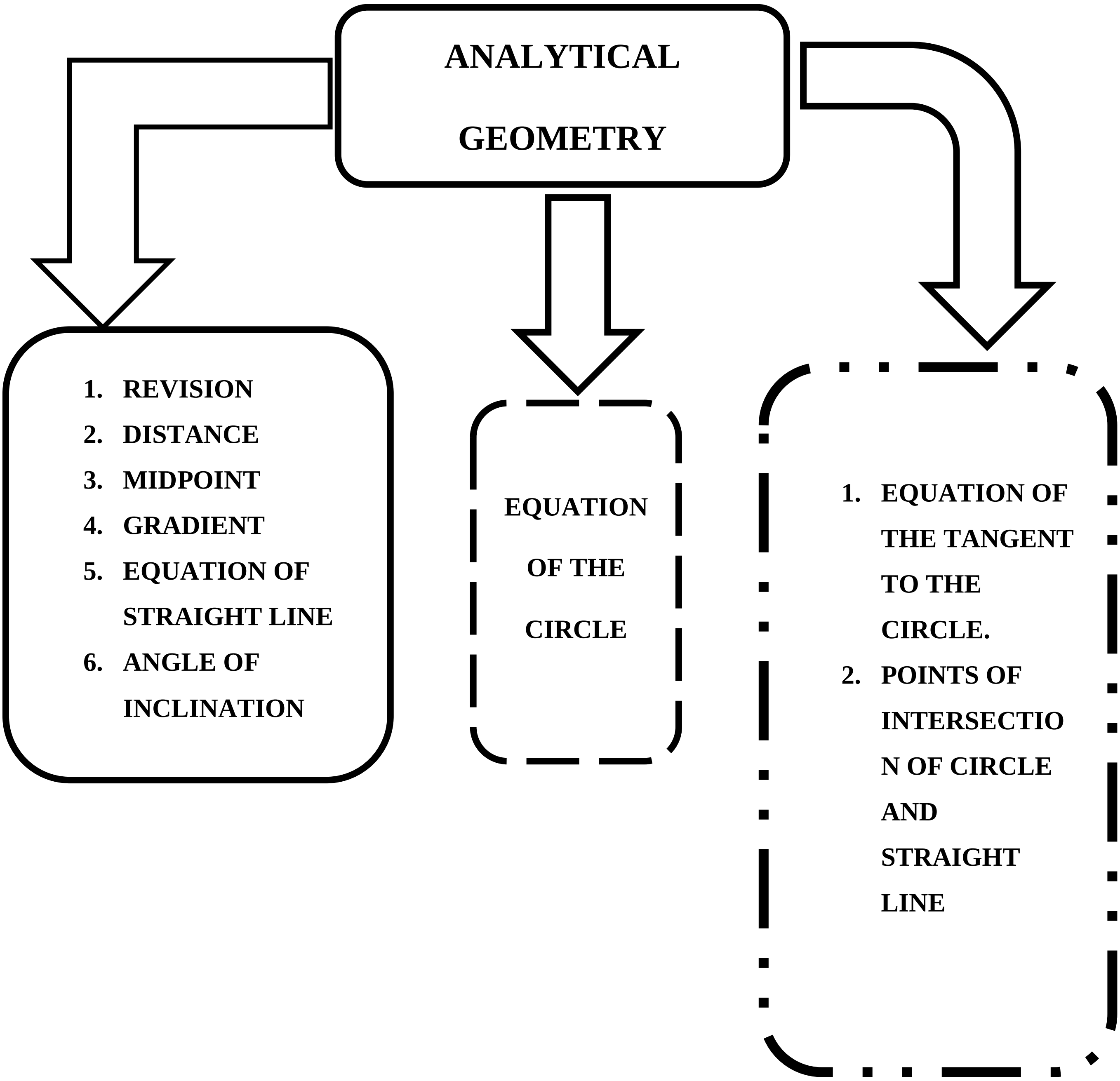
GEOMETRY

BOOKLET

TABLE OF CONTENTS

OVERVIEW

ANALYTICAL GEOMETRY IS THE USE OF FORMULAE TO ANALYSE THE LENGTH, GRADIENT AND OTHER IMPORTANT FEATURES OF CERTAIN POINTS AND THEIR LINES.



ANALYTICAL GEOMETRY GLOSSARY

TERM	EXPLANATION
Distance	Length (in units) from one point to another. Found by using the distance formula using two points given.
Gradient	How steep a line is. Found by using the gradient formula using two points given.
Mid-point	The co-ordinate that represents the middle of a line segment. Found by using the mid-point formula using two points given.
Parallel	Lines that have the same gradient are parallel to each other. Parallel = same gradient
Perpendicular	Two line segments meeting at a right angle.
X - intercept	The point at which a graph cuts the x −axis.
Y - intercept	The point at which a graph cuts the y −axis.
Point of intersection	The co-ordinate where two graphs intersect each other.
Diagonal	The line segment joining opposite corners of a quadrilateral.
Rectangle	A 4-sided shape (quadrilateral) where both pairs of opposite sides are equal in length and all 4 angles are 90°.
Square	A 4-sided shape (quadrilateral) where all 4 sides are equal in length and all 4 angles are 90°.
Kite	A 4-sided shape (quadrilateral) where the adjacent sides (those next to each other) are equal in length. The diagonals are perpendicular to each other.
Rhombus	A 4-sided shape (quadrilateral) is a parallelogram with 4 equal sides.

Parallelogram	A 4-sided shape (quadrilateral) that has 2 pairs of parallel sides.
Equilateral triangle	A triangle with 3 equal sides and 3 equal angles.
Isosceles triangle	A triangle with 2 equal sides and 2 equal angles
Collinear	Points that lie on the same line
Origin	The point where the x and y axis meet on a Cartesian plane.
Line segment	All points between two given points.
Perimeter	The distance around the outside of a shape (the length of the outline of the shape)
Angle of inclination	The angle between a line and the horizontal line (most often the $x - axis$). It can be any measurement from 0° to 180°. It is measured from the horizontal line in an anti-clockwise direction. If the line has a positive gradient, the angle of inclination will be less than 90°. If the line has a negative gradient, the angle of inclination will be between 90° and 180°.
Circle	A curve where all points are the same distance from a given fixed point (the centre).
Circumference	The distance around the circle (the perimeter of the circle).
Equidistant	Exactly the same distance
Radius	The distance from the centre point of a circle to the circumference.
Concentric circles	Circles of different sizes that have a common centre point. (A smaller one would lie inside a larger one)
Tangent	A line that touches a circle at one point only.
Secant	A line that intersects the circle at 2 points.

EXAMINATION GUIDELINES

- Prove the properties of polygons by using analytical methods.
- The concept of collinearity must be understood.
- Candidates are expected to be able to integrate Euclidean Geometry axioms and theorems into Analytical Geometry problems.
- The length of a tangent from a point outside the circle should be calculated.
- Concepts involved with concurrency will not be examined.

ERRORS AND MISCONCEPTIONS ASSOCIATED WITH ANALYTICAL GEOMETRY

- Learners swap the x and y values when substituting into distance, midpoint, and gradient formula.
- Incorrect copying of both distance and gradient formula despite it being given in information sheet.
- Learners forget to use arc tan when asked to find the angle of inclination, θ.
- When calculating the angle of inclination with negative gradient, learners forget to use $180° - \theta$ since the inclination angle will be obtuse i.e. $90° < \theta < 180°$.
- Learners find it difficult to find the equation of a tangent to the circle forgetting that a gradient of a tangent is ALWAYS perpendicular to the gradient of the radius, i.e. $m_1 \times m_2 = -1$.
- Learners often make assumptions about midpoint of line and centre of a circle.

SUGGESTIONS FOR IMPROVEMENT

- The educator must emphasize the use of formula sheet.
- Learners need to be encouraged to label the given coordinates as $(x_1; y_1)$ or $(x_A; y_A)$.
- Emphasize the relationship between the sign of the gradient and the size of the angle of inclination i.e. for $m > 0$, then 0° $< \theta <$ 90° and for $m < 0$, then 90°$< \theta <$ 180° where θ is the angle of inclination.
- No assumptions unless stated otherwise.
- Educator to integrate the analytical geometry with other topics, especially trigonometry and Euclidean geometry.
- Revise the geometry of straight lines:
  - Parallel lines
  - Perpendicular lines
- Learners to be exposed to high order questions (circles touching externally, internally or not touching at all) during informal tasks.

NOTES

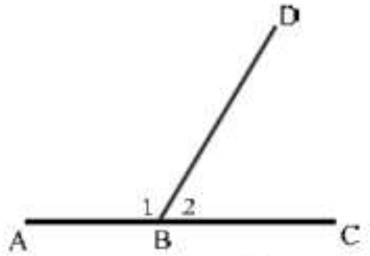
REVISION FROM GRADE 9

GEOMETRY OF STRAIGHT LINES AND ANGLES

Property 1

Adjacent angles on a straight line are supplementary.

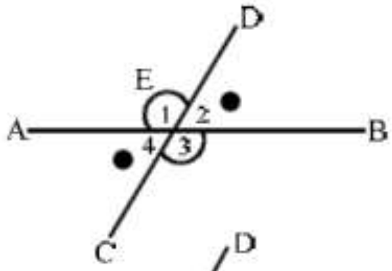
- If ABC is a straight line then $\hat{B}_1 + \hat{B}_2 = 180°$ or
- If $\hat{B}_1 + \hat{B}_2 = 180°$ then ABC is a straight line



Property 2

If two lines AB and CD cut each other (intersect) at E, then the vertically opposite angles are equal.

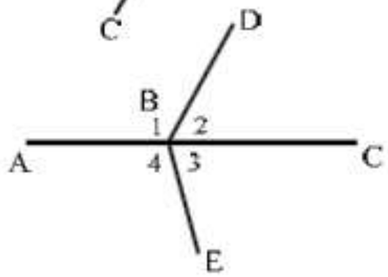
$\hat{E}_1 = \hat{E}_3$ and $\hat{E}_2 = \hat{E}_4$.



Property 3

The angles around a point add up to $360°$.

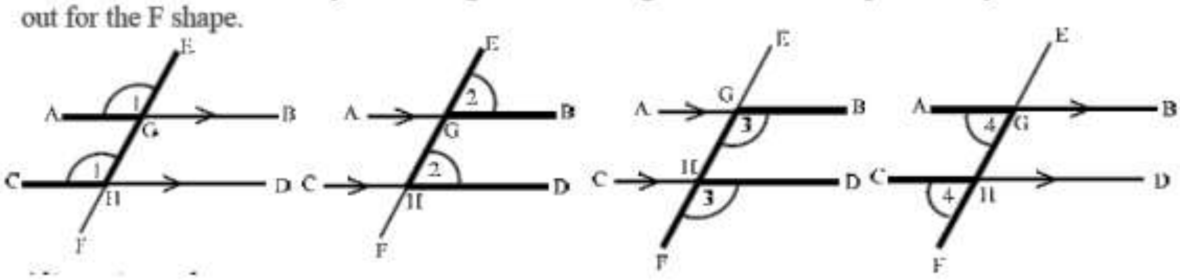
$\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \hat{B}_4 = 360°$



Parallel lines

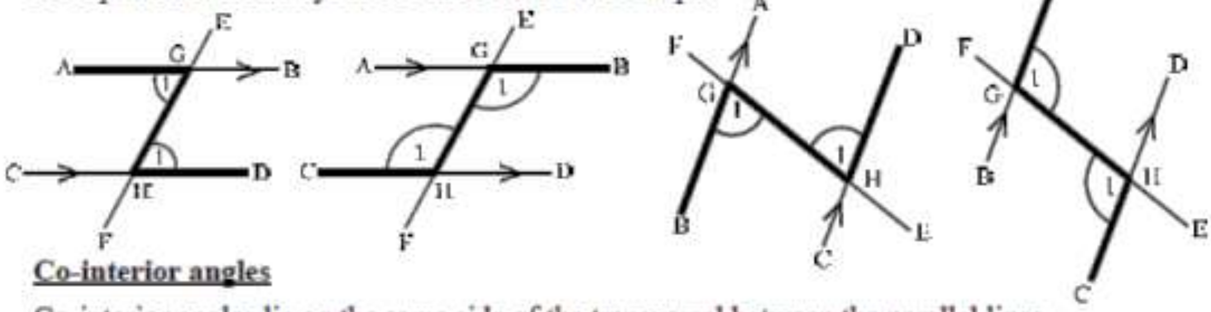
Corresponding angles

Corresponding angles lie either both above or both below the parallel lines and on the same side as the transversal. They are the angles in matching corners and are equal. Always look out for the F shape.



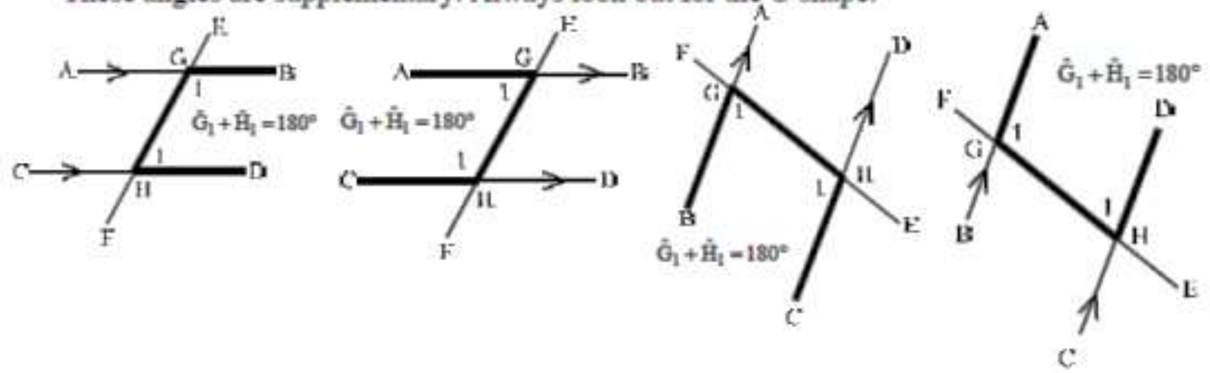
Alternate angles

Alternate angles lie on opposite sides of the transversal and between the parallel lines. They are equal in size. Always look out for the Z or N shape.



Co-interior angles

Co-interior angles lie on the same side of the transversal between the parallel lines. These angles are supplementary. Always look out for the U shape.



LINES AND POLYGONS

a. **Distance between two points**

To find the distance of any point: use the formula

$$d = \sqrt{(x_B - y_B)^2 + (x_A - y_A)^2}$$

Example: Find the distance from the point A(-2; 3) to point B (3; -2)

Answer: $\therefore\ d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

$$\therefore d = \sqrt{\left(3 - (-2)\right)^2 + (-2 - 3)^2}$$
$$\therefore d = \sqrt{(5)^2 + (-5)^2}$$
$$\therefore d = \sqrt{25 + 25}$$
$$\therefore d = \sqrt{50}$$
$$\therefore d = 5\sqrt{2}$$

b. **Midpoint of a line.**

To find the midpoint of two coordinates you have to find the "average" of the two x-coordinates, and find the average of the two y-coordinates.

The formula to find the midpoint is:

$$M\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$$

Example: Find the midpoint of the points A (2; 5) and B (-3; 8)

$$M_{AB}=\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$$
$$\therefore M_{AB}=\left(\frac{2-3}{2};\frac{5+8}{2}\right)$$
$$\therefore M_{AB}=\left(-\frac{1}{2};6\frac{1}{2}\right)$$

c. **Gradient of a line**

The gradient works out the slope of the graph – or the ratio of the height to the length – between two points. The formula for the gradient is: $m=\frac{y_2-y_1}{x_2-x_1}$

Example – Find the gradient between two points A $\left(\frac{1}{4};-2\right)$ and B (2; 1)

$$\therefore m=\frac{y_2-y_1}{x_2-x_1}$$
$$\therefore m=\frac{1-(-2)}{2-\left(\frac{1}{4}\right)}$$
$$\therefore m=\frac{12}{7}$$

In other words – for every 12 units you go up, you move 7 units right.

d. **Parallel and Perpendicular Gradients**

- When one line is parallel to another line, the gradients are equal: i.e. $m_1=m_2$.
- When one line is perpendicular (at a right angle) to another line, the gradients are the negative inverse of each other. This means that if you multiply the first gradient with the second gradient it will give you -1. i.e. $m_1\times m_2=-1$.

e. **EQUATIONS OF STRAIGHT LINES.**

The straight-line equation is given by: $y = mx + c$ and $y - y_1 = m(x - x_1)$ Where:

• y → depends on your x-value.

• m → gradient

• x → determines your y-value.

• c → your y-intercept (where the straight line crosses the y-axis).

To find the equation of a straight line:

Scenario 1: Two points given

When you are given two coordinates first work out the gradient, and then substitute it into the formula for the straight line, along with one of your coordinates. Solve for c Now write down the equation with your values for m and c.

Example 1.

Find the equation of the straight-line DE if D (1; -7) and E (-3; 1).

Step 1: find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore m = \frac{1-(-7)}{-3-1}$$
$$\therefore m = -2$$

Step 2: find the value of c.

$y = mx + c$ Substitute $m = -2$ and (1; -7)

$$-7 = -2(1) + c$$
$$-5 = c$$

Step 3: finally, write down the equation in standard form.

$\therefore y = -2x - 5$

Scenario 2: Given the gradient is parallel or perpendicular to another line and a point:

Determine if your gradient is the same (**parallel lines**) or the negative inverse of the other gradient (**perpendicular lines**). Next substitute your coordinate to find c. Finally, write down your equation in standard form.

Example 2.

Determine the equation of the straight line that is parallel to the line $y = 3x + 4$ passing through the point (-1 ; 3).

Step 1: determine the gradient.

From the parallel lines, we can see that the gradients are equal or $m = 3$.

Step 2: Now find the c value by first substituting in the value for m, $x - value$ and $y - value$ from the coordinates.

$y = 3x + c$ Substitute in the coordinate (-1; 3)

$3 = 3(-1) + c$
$6 = c$

Step 3: finally, write the equation in standard form.

$\therefore y = 3x + 6$

Activity 1: Level 1 and 2.

a) the line passes through the point K (-1; 2) and is parallel to the line $y = 2x + 1$.

b) the line passes through the point L (3; 8) and is perpendicular to the line $y = -4x - 2$

c) the line passes through the point M $\left(9; -\frac{1}{2}\right)$ and is perpendicular to the line $y = 6x - \frac{5}{6}$

d) the line passes through the point N $\left(-\frac{1}{5};\ 9\right)$ and is parallel to the line $y = -\frac{1}{3}x - 4$

e) the line passes through the point P (4; -6) and is perpendicular to the line $y = 5x - 4$

f. **Angle of inclination**

The angle of inclination tells you at what angle the straight line crosses the x-axis. To work out the angle of inclination first find your gradient, then substitute it into the formula and solve for the angle using this formula:

$$\tan\theta = m$$

NB: When your gradient is positive, you can simply use the angle given when you work it out. However, if your gradient is NEGATIVE then work out the angle using a positive value of the gradient (in other words, multiply your gradient by negative 1) and then subtract the angle from 180°.

Example 3. Positive gradient

Determine the angle of inclination for the line $y = 3x - 1$.

$m = 3$

$\therefore tan\theta = 3$

$\theta = 72°$

Negative gradient:

Determine the angle of inclination for the line $y = -\frac{1}{4}x - 6$

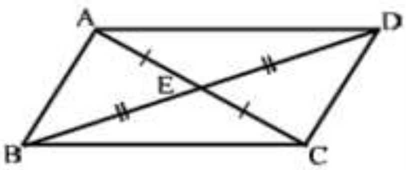
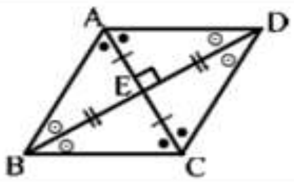
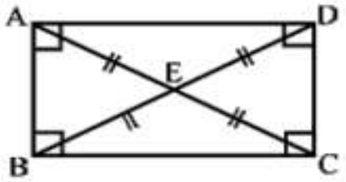
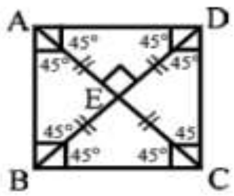
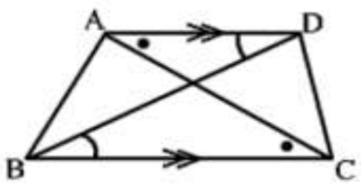
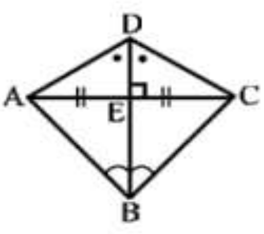
$m = \frac{-1}{4}$, the positive value is $\frac{1}{4}$

$tan\theta = \frac{1}{4}$

$\therefore \theta = 14°$. So, angle of inclination $= 180° - 14° = 166°$.

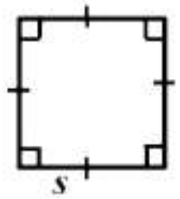
PROPERTIES OF POLYGONS

In Analytical Geometry questions can be asked which require you to know the properties of the various quadrilaterals. The properties of the various quadrilaterals you have learnt in grade 10 and is given in the table below. The table below captures the various properties you have to show a particular quadrilateral has for it to be a Parallelogram, Rectangle, Rhombus, Square, Trapezium or Kite.

Quadrilateral	Diagonals	Angles	Sides
	The diagonals of a parallelogram bisect each other.	The opposite angles of a parallelogram are equal. The interior angles add up to $360°$.	The opposite sides of a parallelogram are parallel and equal.
	The diagonals of a rhombus bisect each other at right angles. The diagonals bisect the vertex angles.	The opposite angles of a rhombus are equal. The interior angles add up to $360°$.	The opposite sides of a rhombus are parallel and all sides are equal.
	The diagonals of a rectangle bisect each other and are equal in length.	The interior angles of a rectangle are equal to $90°$. The interior angles add up to $360°$.	The opposite sides of a rectangle are parallel and equal.
	The diagonals of a square bisect each other at right angles and are equal in length. The diagonals bisect the vertex angles.	The interior angles of a square are equal to $90°$. The interior angles add up to $360°$.	The opposite sides of a square are parallel and all sides are equal.
	The diagonals of a trapezium intersect but don't bisect each other. They lie between parallel lines and therefore the alternate angles are equal.	The interior angles add up to $360°$.	One pair of opposite sides are parallel.
	The diagonals are perpendicular and one diagonal bisects the other. One of the diagonals bisects the vertex angles.	One pair of opposite angles are equal. The interior angles add up to $360°$.	Two pairs of adjacent sides are equal.

AREA AND PERIMETER OF POLYGONS

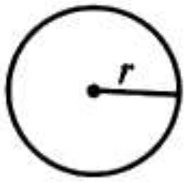
Square



Per.= $4 \times s$

Area = $s \times s = s^2$

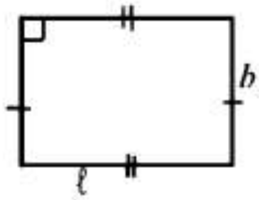
Circle



Circumf.= $2\pi r$

Area = πr^2

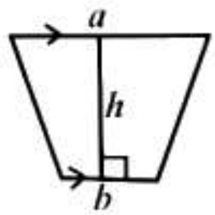
Rectangle



Per.= $2\ell + 2b$

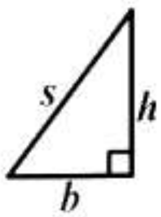
Area = $\ell \times b$

Trapezium



Area = $\frac{1}{2}(a + b) \times \perp h$

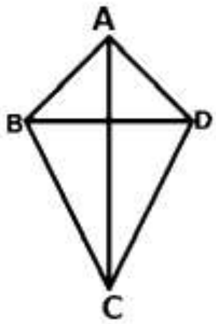
Triangle



Per.= $b + s + h$

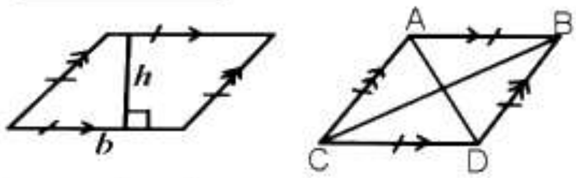
Area = $\frac{1}{2}\, b \times \perp h$

Kite



Area = ½(product of diagonals)
= ½(AC x BD)

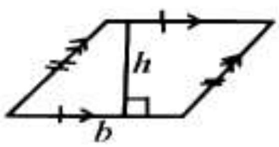
Rhombus



Area = $b \times h$ or

Area = ½(product of diagonals)
= ½(AD x BC)

Parallelogram

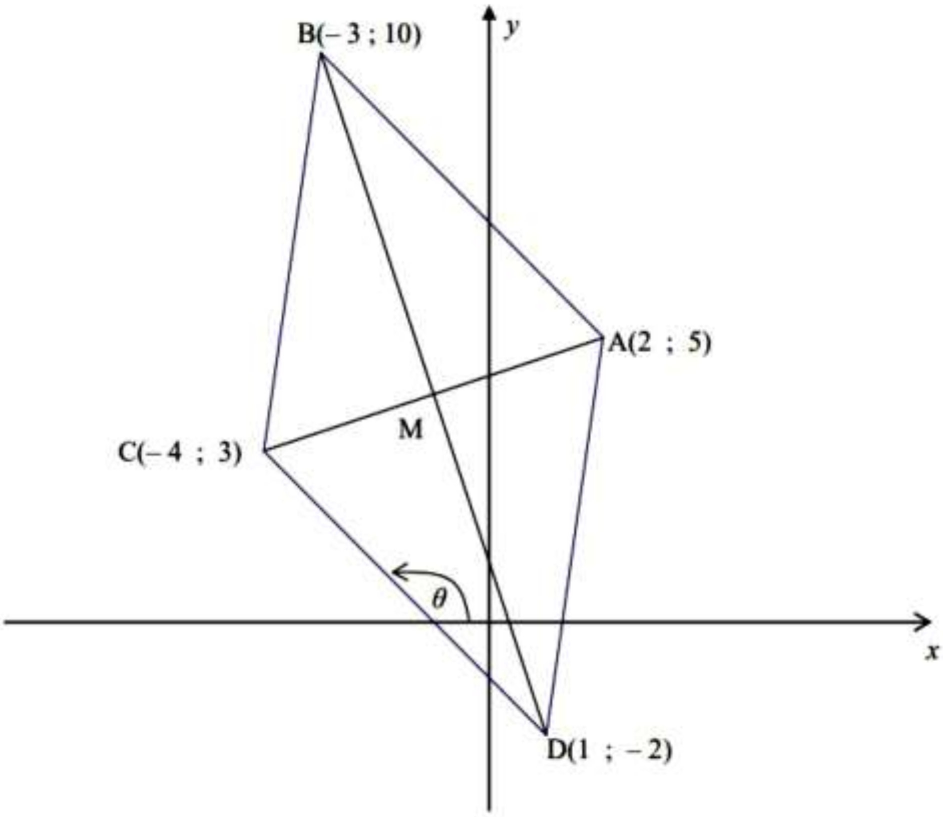


Area = $b \times h$

EXAM-BASED QUESTIONS OF COGNITIVE LEVEL 1 AND 2: LINES AND POLYGONS

QUESTION 1

$ABCD$ is a quadrilateral with vertices $A(2; 5), B(-3; 10), C(-4; 3)$and $D(1; -2)$.

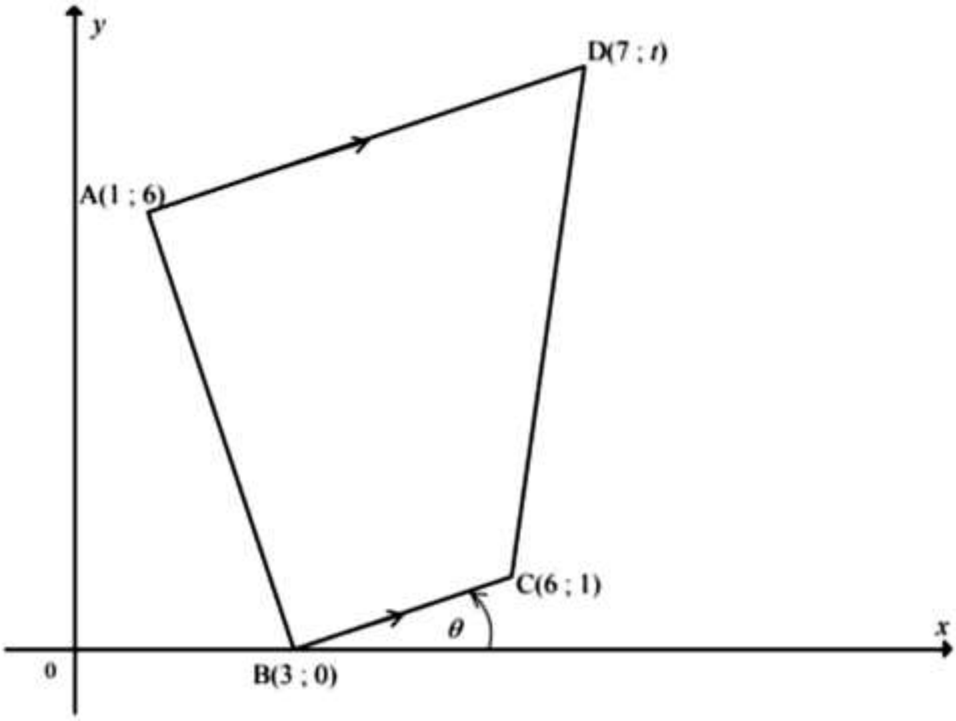


1.1.	Calculate the length AC. (Leave the answer in simplest surd form).	(2)
1.2.	Determine the coordinates of M, the midpoint of AC.	(2)
1.3.	Show that BD and AC bisect each other at right angles at M.	(5)
1.4.	Calculate the area of ΔABC.	(4)
1.5.	Determine the equation of DC.	(3)
1.6.	Determine θ, the angle of inclination of DC.	(2)

[18]

QUESTION 2

$ABCD$ is a quadrilateral with vertices $A(1;6), B(3;0), C(6;1)$ and $D(7:t)$ in a Cartesian plane $AD \parallel BC$.



2.1. Calculate the gradient of BC. (2)

2.2. Determine the equation of AD in the form $y = \cdots$ (3)

2.3. Show that $t = 8$. (2)

2.4. Calculate the lengths of AD, BC and AB. (4)

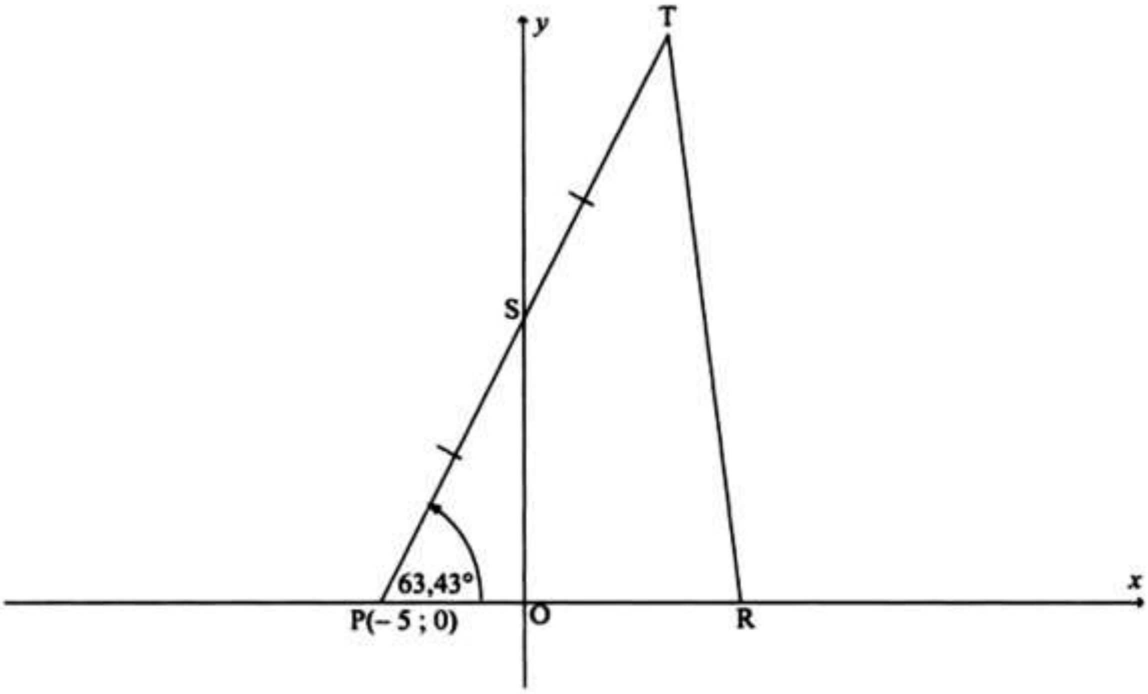
2.5. Show that AB is perpendicular to BC. (3)

2.6. Determine θ, the angle of inclination of BC. (3)

[17]

QUESTION 3

In the diagram below, P is a point $(-5; 0)$. The inclination of line PT is $63{,}43°$. S is the midpoint and the y-intercept of PT. R is a point on the x-axis such that $PO : OR = 2 : 3$.



3.1. Determine:

3.1.1. The gradient of PT, correct to the nearest integer value. (2)

3.1.2. The equation of PT in the form $y = mx + c$. (2)

3.1.3. The distance PS in surd form. (3)

3.1.4. The coordinates of T. (2)

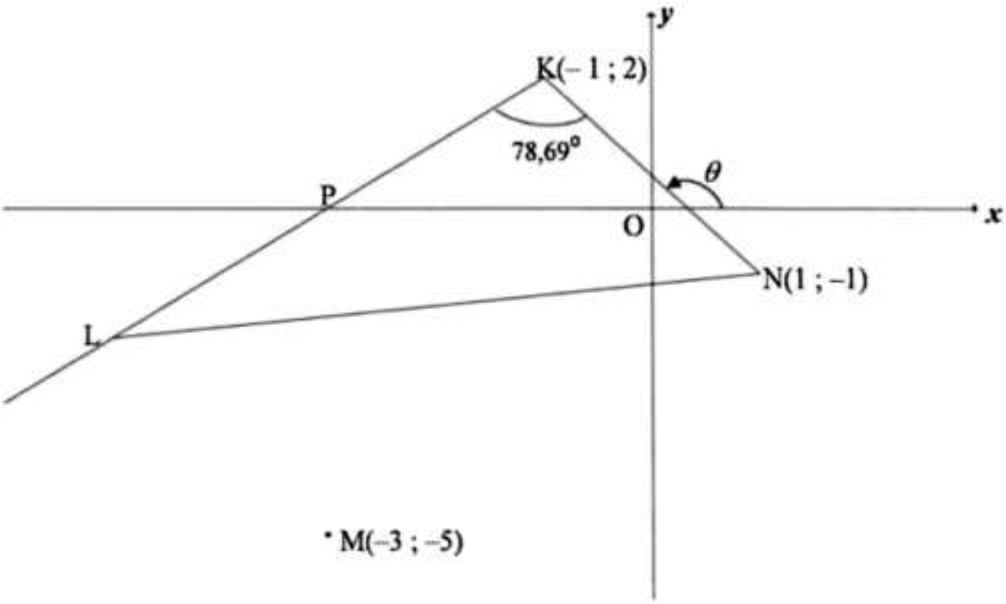
[9]

QUESTION 4

In the diagram, $K(-1;2), L$ and $N(1;-1)$ are vertices of ΔKLN such that $L\widehat{K}N = 78{,}69°$.

KL intersects the x-axis at P. KL is produced. The inclination of KN is θ.

The coordinates of M are (-3;-5).



4.1. Calculate:

4.1.1. The gradient of KN. (2)

4.1.2. The size of θ, the inclination of KN. (2)

4.2. Show that the gradient of KL is 1. (2)

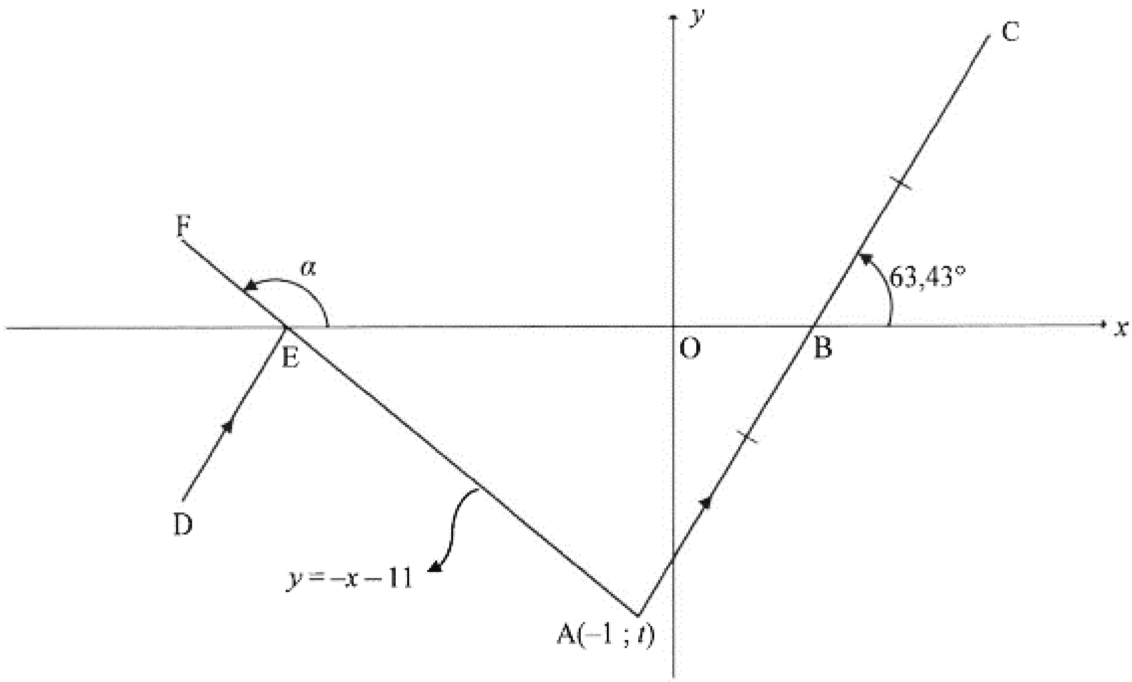
4.3. Determine the equation of the straight line KL in the form $y = mx + c$. (2)

4.4. Calculate the length of KN. (2)

[10]

QUESTION 5

In the diagram, the equation of line AF is $y = -x - 11$. B, a point on the x-axis is the midpoint of the straight line joining $A(-1; t)$ and C. The angles of inclination of AF and AC are α and $63{,}43°$ respectively. AF cuts the x-axis in E. d is a point such that DE∥AC.



5.1 Calculate the:

5.1.1 Value of t (2)

5.1.2 Size of α (2)

5.1.3 Gradient of AC, to the nearest whole number (2)

5.2 Determine the equation of AC in the form $y = mx + c$ (2)

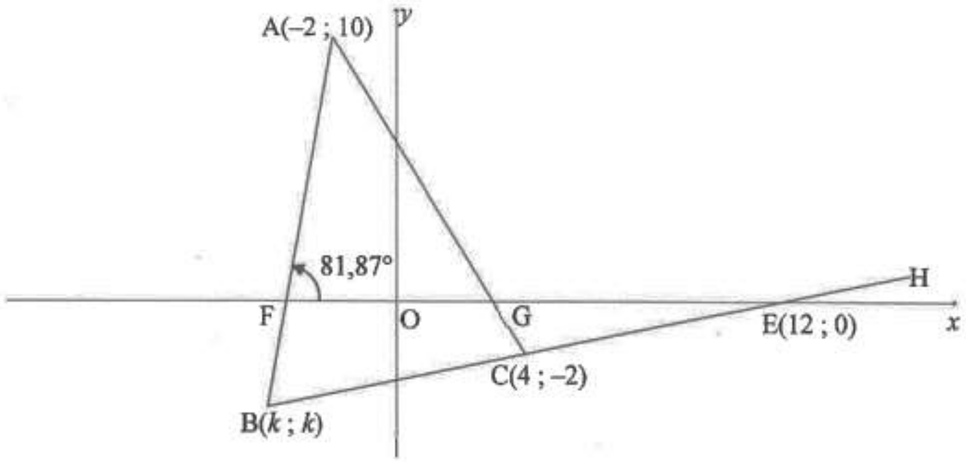
5.3 Calculate the:

5.3.1 Coordinates of C (3)

[11]

QUESTION 6

In the diagram, $A(-2; 10), B(k; k)$ and $C(4; ; -2)$ are the vertices of ΔABC. Line BC is produced to H and cuts the x-axis at $E(12; 0)$. AB and AC intersects the x-axis at F and G respectively. The angle of inclination of line AB is81,87°.



6.1. Calculate the gradient of:

6.1.1. BE (2)

6.1.2. AB (2)

6.2. Determine the equation of BE in the form $y = mx + c$ (2)

6.3. Calculate:

6.3.1. Coordinates of B, where $k < 0$ (2)

6.3.2. Coordinates of the point of intersection of the diagonals of parallelogram ACES, where S is a point in the first quadrant. (2)

6.4. Another point $T(p; p)$, where $p > 0$, is plotted such that $ET = BE = 4\sqrt{17}$ units.

6.4.1. Determine the equation of T.

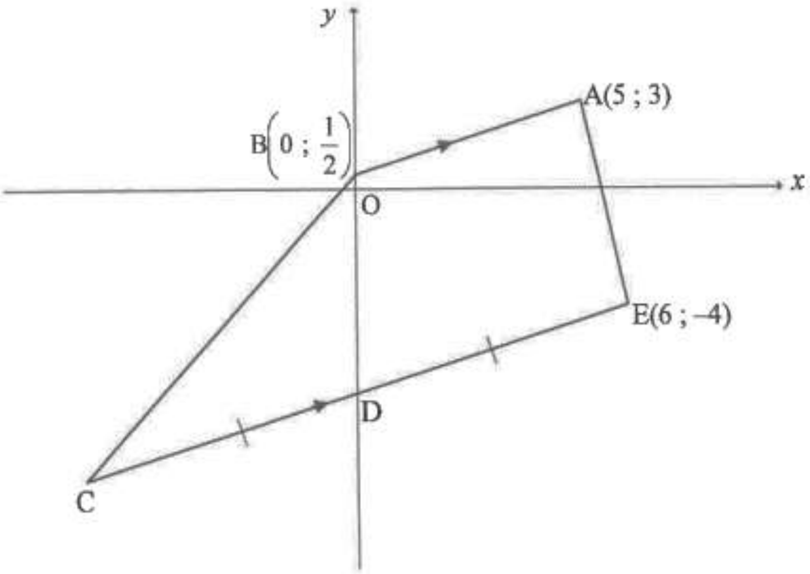
a) Circle with centre at E and passing through B and T in the form $(x-a)^2 + (y-b)^2 = r^2$ (2)

b) Tangents to the circle at point $B(k; k)$ (3)

[15]

QUESTION 7

In the diagram, $A(5;3), B\left(0;\frac{1}{2}\right), C$ and $E(6;-4)$ are the vertices of a trapezium having $BA \parallel CE$. D is the y-intercept of CE and $CD = DE$.



7.1 Calculate the gradient of AB (2)

7.2 Determine the equation of CE in the form $y = mx = c$ (3)

7.3 Calculate the:

7.3.1 Coordinates of C (3)

7.4 If point K is the reflection of E in the y-axis:

7.4.1 Write down the coordinates of K (2)

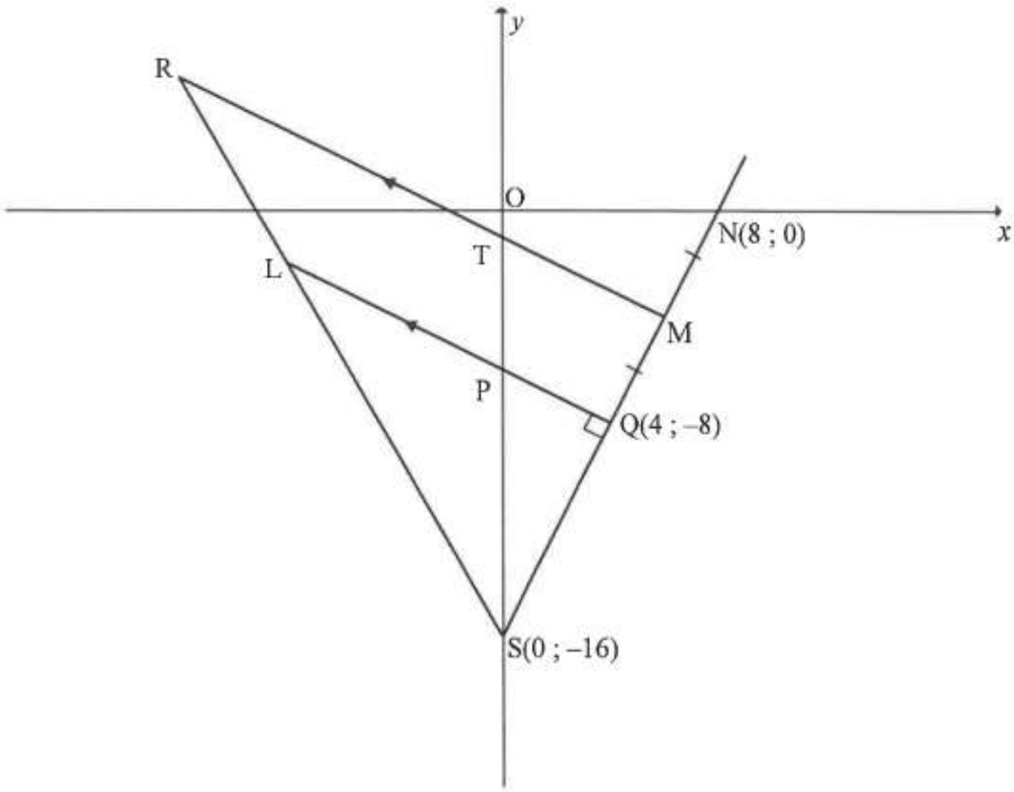
7.4.2 Calculate the:

(a) Perimeter of ΔKEC (4)

[14]

QUESTION 8

In the diagram, $S(0;-16)$, L and $Q(4;-8)$ are the vertices of ΔSLQ having LQ perpendicular to SQ. SL and SQ are produced to points R and M respectively such that RM ∥ LQ. SM produced cuts the x-axis at N $(8;0)$. QM = MN. T and P are the y-intercepts of RM and LQ respectively.



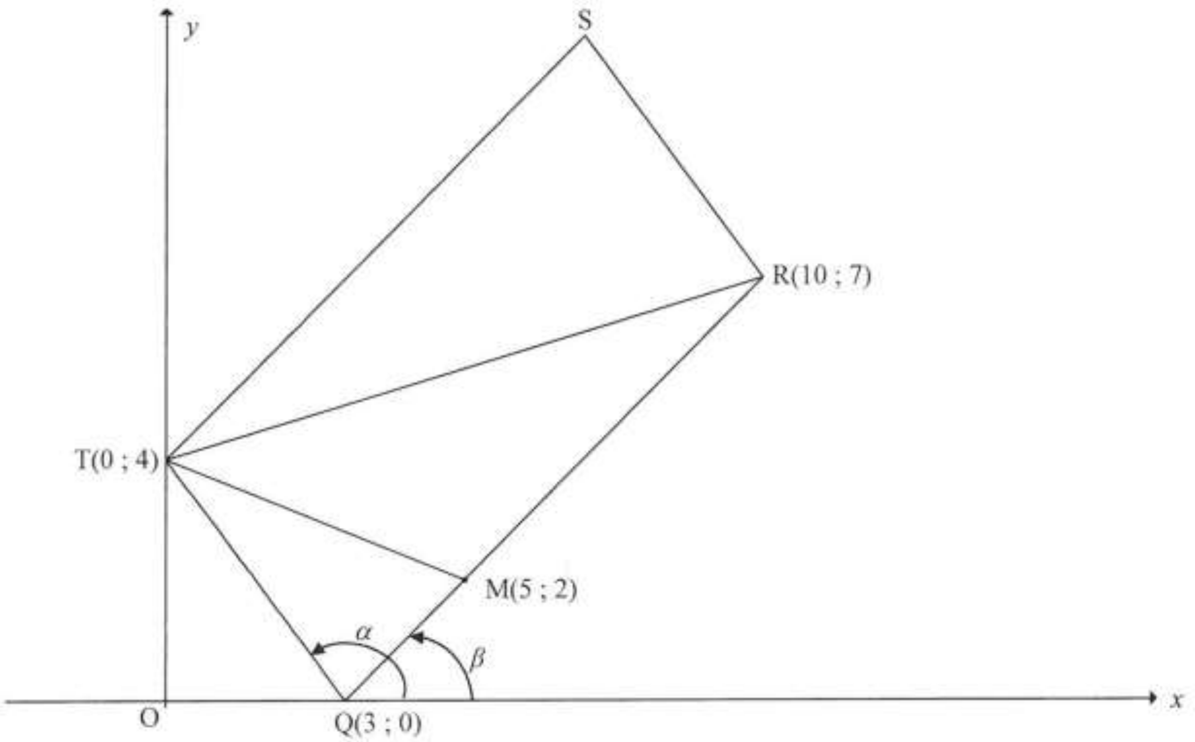
8.1	Calculate the coordinates of M.	(2)
8.2	Calculate the gradient of NS.	(2)
8.3	Show that the equation of line LQ is $y=\frac{-1}{2}x-6$.	(3)
8.5	Calculate the coordinates of T.	(3)

[10]

QUESTION 9

In the diagram, $Q(3;0), R(10;7), S$ and $T(0;4)$ are the vertices of parallelogram QRST.

From T a straight line is drawn to meet QR at $M(5;2)$. The angles of inclination of TQ and RQ are α and β respectively.



9.1. Calculate the gradient of TQ. (1)

9.2. Calculate the length of RQ. Leave your answer in surd form. (2)

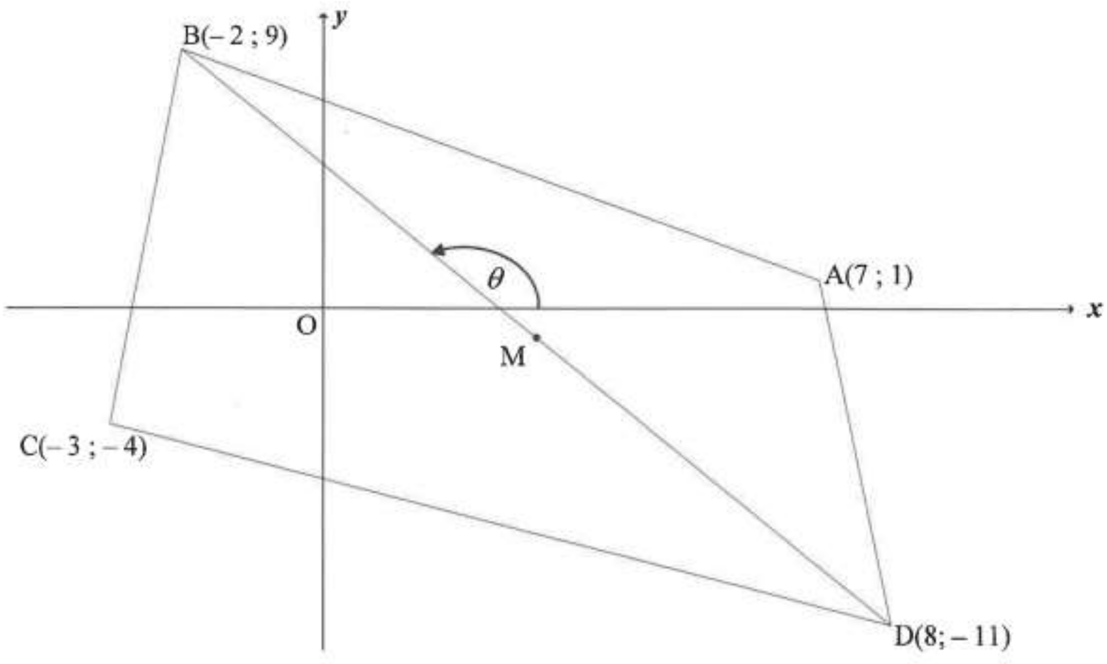
9.3. $F(k;-8)$ is a point in the Cartesian plane such that T, Q and F are collinear.

Calculate the value of k. (4)

[7]

QUESTION 10

In the diagram, $ABCD$ is a parallelogram having vertices $A(7;1), B(-2;9), C(-3:-4)$ and $D(8;-11)$. M is the midpoint of BD.



10.1. Calculate the gradient of AC. (2)

10.2. Determine:

10.2.1. The equations of AC in the form $y = mx + c$ (2)

10.2.2. Whether M lies on AC (4)

10.3. Prove that $BD \perp AC$. (3)

10.4. Calculate:

10.4.1. θ, the inclination of BD. (2)

10.4.2. The length of AC. (2)

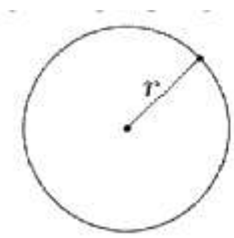
[15]

PART B: CIRCLES

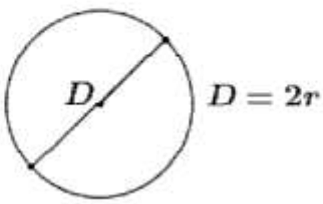
A **circle** is a set of points which are equidistant from a fixed point, the centre of the circle. The distance from the centre to any point on the circle is the length of the radius while the distance around the whole circle is the circumference or the perimeter of the circle.

Terminology related to circles:

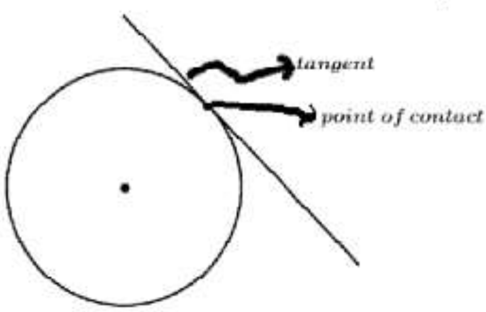
- **Radius**: any line segment that connects the centre of the circle and any point on the circumference.



- **Diameter**: any line segment that joins two points on the circumference and passes through the centre.



- **Tangent**: Any line segment that touches a circle externally at **ONLY** one point.



1. EQUATION OF CIRCLE.

The standard form of equation of a circle at **ANY** centre is given by:

$(x-a)^2+(y-b)^2=r^2$ where $(a;b)$ is the centre of the circle, r is the radius of the circle and x and y are coordinates of a point on the circle.

NB!!!

To find the radius of the circle, remove the square-by-square root. Since the radius is a length, ONLY consider the positive answer.

Circle with centre at the origin (0 ; 0),

$$(x-0)^2+(y-0)^2=r^2$$

$$\therefore x^2+y^2=r^2$$

The general form of the equation of the circle is $ax^2+by^2+cx+dy+e=0$.

To switch from general form to standard form, you have to **COMPLETE THE SQUARE**.

Example 1

Determine the centre of the circle and radius of the following circles

a. $(x-2)^2+(y+3)^2=25$

b. $x^2+y^2-x-2y-5=0$

Solutions

a. Centre $(2;-3)$, $r=\sqrt{25}=5$

b. Step 1:

Rewrite the equation $x^2 - x + y^2 - 2y = 5$. The x and y terms are written separately and the constant term is moved to the right hand side of the equation.

Step 2:

Halve the co-efficient of x and add the square of the result on both sides of the equation. Repeat the same process for y.

$$x^2 - 6x + (-3)^2 + y^2 + 2y + (1)^2 = -8 + 9 + 1$$

Step 3

Factorise: $(x-3)^2 + (y+1)^2 = 2$

Centre $(3; -1)$, $r = \sqrt{2}$

2. Finding the equation of a circle.

Scenario 1: If you know the centre and one point on the circle.

Centre (-4; 2) and point (-2; 6)

Step 1: substitute the centre points i.e. $(x+4)^2 + (y-2)^2 = r^2$

Step 2: substitute a point to get radius squared, $(-2+4)^2 + (6-2)^2 = r^2 = 20$

Step 3: finally, write the equation: $(x+4)^2 + (y-2)^2 = 20$

Scenario 2: If you know the centre and the radius.

Centre (3; -3) and radius 4

Step 1: substitute the centre points i.e. $(x-3)^2 + (y+3)^2 = r^2$

Step 2: determine radius squared, $r^2 = 4^2 = 16$

Step 3: finally, write the equation: $(x-3)^2 + (y+3)^2 = 16$

Scenario 3: If you know the end-points of a diameter

Given A (2; -3) and B (6; -1), AB is a diameter of a circle.

Step 1: determine the centre using the midpoint formula and substitute into the equation.

Centre $\left(\frac{2+6}{2}; \frac{-3-1}{2}\right)$, $\therefore$ centre is $(4;-2)$, $(x-4)^2+(y+2)^2=r^2$

Step 2: Use one point to find radius squared

$(2-4)^2+(-3+2)^2=r^2=5$

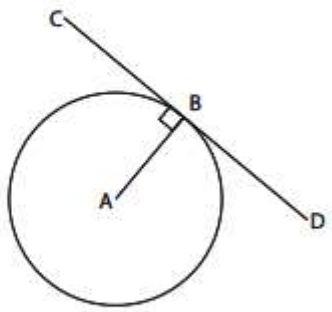
Step 3: finally, write the equation: $(x-4)^2+(y+2)^2=5$

3. EQUATION OF THE TANGENT TO THE CIRCLE

A **tangent** is a straight line that is drawn perpendicular to the circle's radius and touching the circle at only one point.

To work out the equation of the tangent use the straight-line formula: $y=mx+c$.

In the diagram alongside CBD is a tangent to the circle with centre A.



REMEMBER: Tangent is perpendicular to the radius. That means the angle between a tangent and a radius is 90°. This means that their gradients are negative reciprocals of each other.

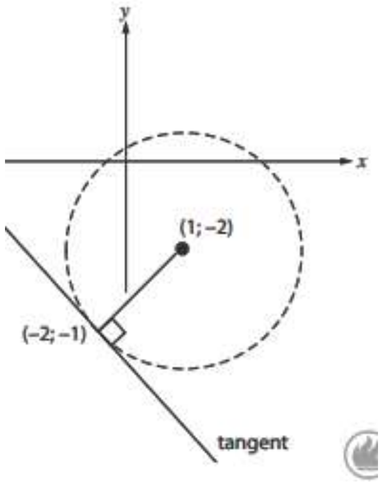
In order to find the equation of a tangent it is important to know that:

$m_{radius} \times m_{tangent}=-1$.

Consider the following example:

Determine the equation of the tangent to the circle $(x-1)^2+(y+2)^2=10$ at the point $(-2;-1)$

Step 1: Write down the co-ordinates of the centre of the circle and draw a rough diagram.



Step 2: calculate the gradient of the radius.

$m_{radius}=\frac{-2-(-1)}{1-(-2)}=-\frac{1}{3}$

Step 3: Determine the gradient of the tangent using $m_{radius}\times m_{tangent}=-1$

$m_{radius}=-\frac{1}{3}$

$\therefore m_{tangent}=3$

Step 4: use $y-y_1=m(x-x_1)$ and the point of contact to find the equation of the tangent.

$y-(-1)=3\big(x-(-2)\big)$

$\therefore y=3x+5$

NOTE: To find equation of a tangent

- Use coordinates of centre and point of tangency (point of contact) to get gradient of a radius. If the centre is not given, use the end points of a diameter to get the gradient of the diameter
- Use $m_{radius} \times m_{tangent} = -1$, to get the gradient of tangent.
- Use $y - y_1 = m(x - x_1)$ and the point of contact to find the equation of the tangent.

BOOM: you have your tangent line equation!

4. LENGTH OF A TANGENT

Let us have a look at how we use the skills we have learnt so far to calculate the length of a tangent.

Consider the following example:

Determine the length of the section of the tangent drawn from **(6; –2)** to point of intersection with the circle $x^2 - 6x + y^2 + 2y + 8 = 0$.

Step 1: First we will write the equation of our circle in the form $(x - a)^2 + (y - b)^2 = r^2$ by completing the square and the radius.

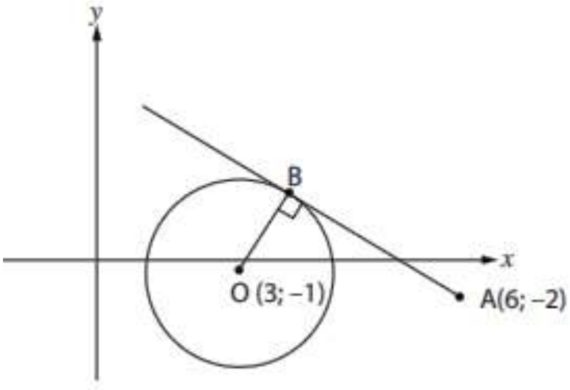
$\therefore x^2 - 6x + 9 + y^2 + 2y + 1 = 0 + 9 - 8 + 1$

$(x - 3)^2 + (y + 1)^2 = 2$

Centre (3; -1)

$r = \sqrt{2}$

Step 2: Draw the rough diagram



From the diagram, it clearly shows that the length of a tangent is segment AB.

Step 3: Determine distance from the centre of a circle to point A, i.e. point where a section of a tangent starts.

OA $= \sqrt{(3-6)^2 + (-1-9-2))^2}$

OA $= \sqrt{10}$

Step 4: finally, determine the segment AB using Pythagoras theorem. Since we know that B = 90° (radius perpendicular to tangent).

We know the OB $= \sqrt{2}$ units since is the radius of the circle.

$OA^2 = OB^2 + AB^2$

$(\sqrt{10}\,)^2 = (\sqrt{2})^2 + AB^2$

$\therefore AB = \sqrt{8}$

NOTE: How to find the length of tangent

- Determine the centre of the circle,
- Use distance formula to find the length from centre to the point where tangent starts
- Finally, use Pythagoras theorem to find length of a tangent.

RELATIONSHIP BETWEEN THE RADII AND THE DISTANCE BETWEEN THE CENTRES OF EACH CIRCLE.

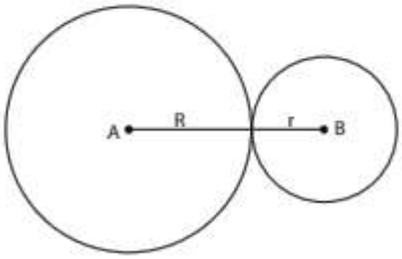
Two circles can touch once, twice or not touch at all.

i. Touches each other once (internally and externally).

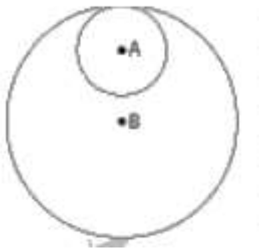
How to prove that the two circles touch each other?

Let the centre of the one circle be A radius be R and the centre of another circle be B and radius be r. Let the distance between the two centres be AB.

- Calculate the distance AB using the distance formula, then add R (the radius of the one circle) to r the radius of the other.
- If $AB = R + r$, then the two circles touch each other externally

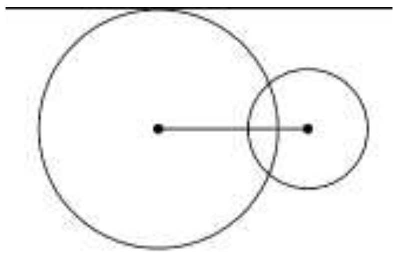


- If $AB = R - r$, then the two circles touch each other internally.



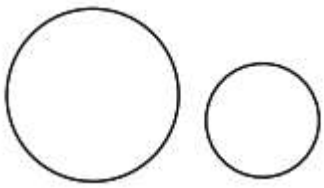
ii. Intersect each other

- If $AB < R + r$, then the circles generally intersect at two points.



iii. Do not touch each other.

If $AB > R + r$, then the two circles never touch.



EXAM-BASED QUESTIONS OF COGNITIVE LEVEL 1 AND 2: CIRCLES

QUESTION 1

1.1 Determine the centre and radius of the circle with the equation

$x^2 + y^2 + 8x + 4y - 38 = 0$ (4)

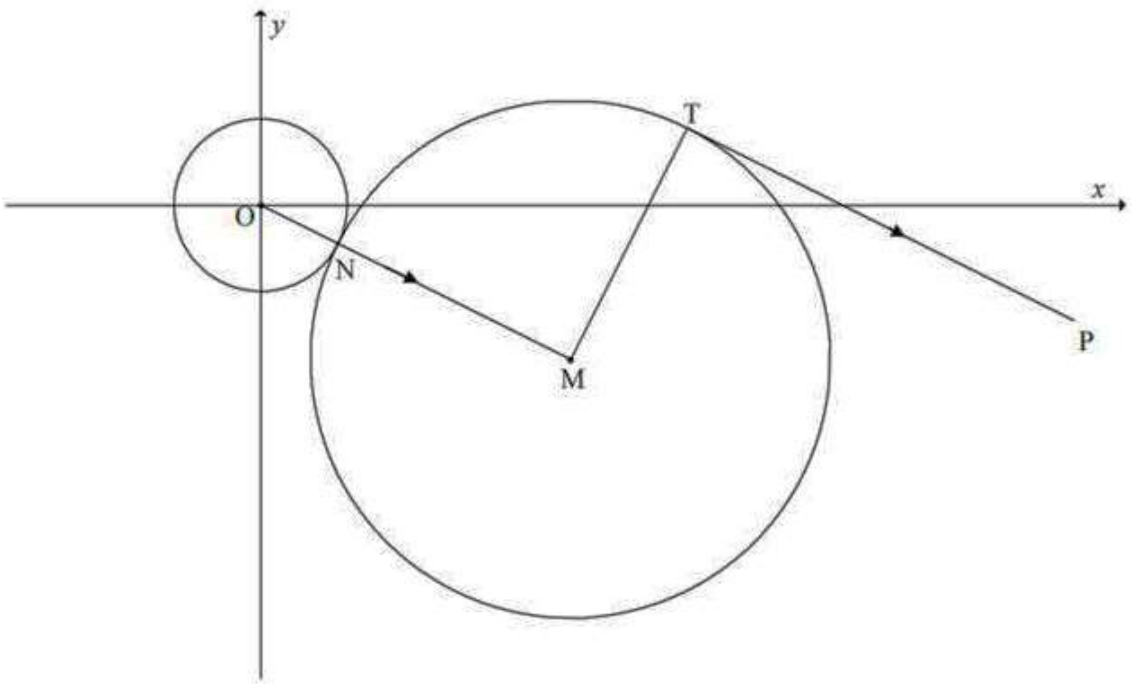
1.2 A second circle has the equation $(x - 4)^2 + (y - 6)^2 = 26$. Calculate the distance between the centres of the two circles. (2)

[6]

QUESTION 2

In the diagram below, the equation of the circle with centre M is $(x - 8)^2 + (y + 4)^2 = 45$.

PT is a tangent to this circle at T and PT is parallel to OM. Another circle having centre O, touches the circles having centre M at N.



2.1 Write down the coordinates of M. (1)

2.2 Calculate the length of OM. Leave your answer in simplest form. (2)

2.3 Calculate the length of ON. Leave your answer in simplest form. (3)

2.4 Calculate the size of $O\hat{M}T$. (2)

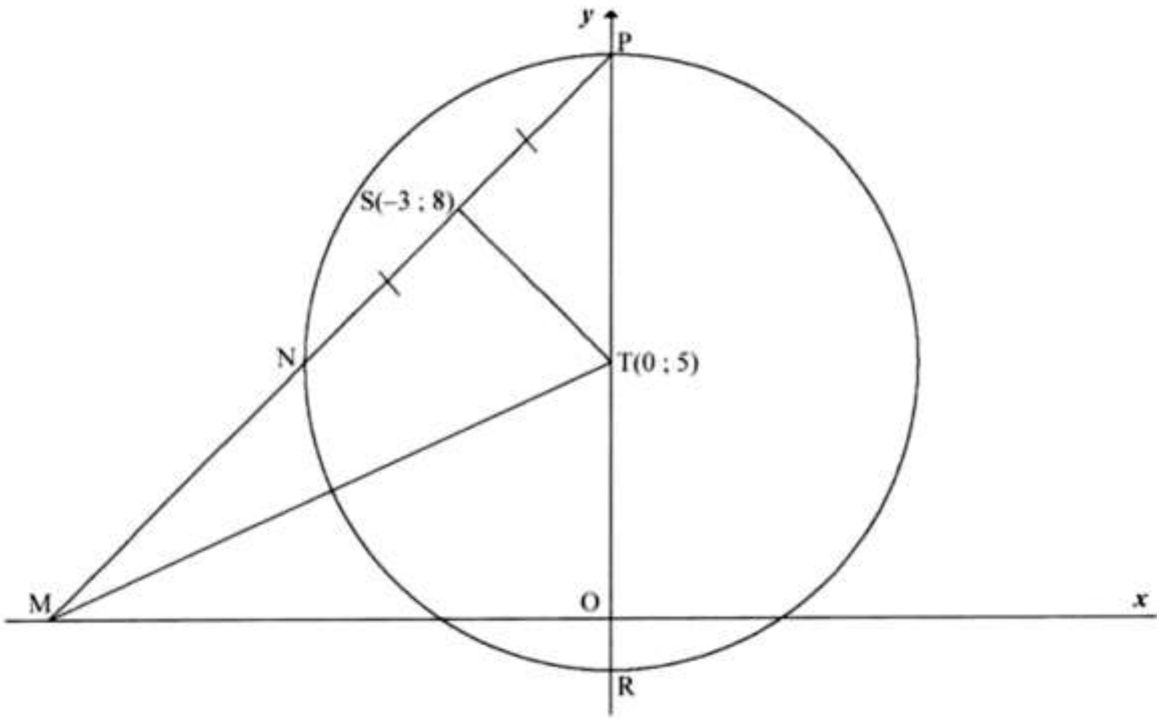
2.5 Determine the equation of MT in the form $y = mx = c$. (5)

[13]

QUESTION 3

In the diagram, the circle, having centre $T(0; 5)$, cuts the y-axis at P qnd R. The line through P and $S(-3; 8)$ intersects the circle at N and the x-axis at M. NS =PS.

MT is drawn.



3.1 Give a reason why TS ⊥ NP. (1)

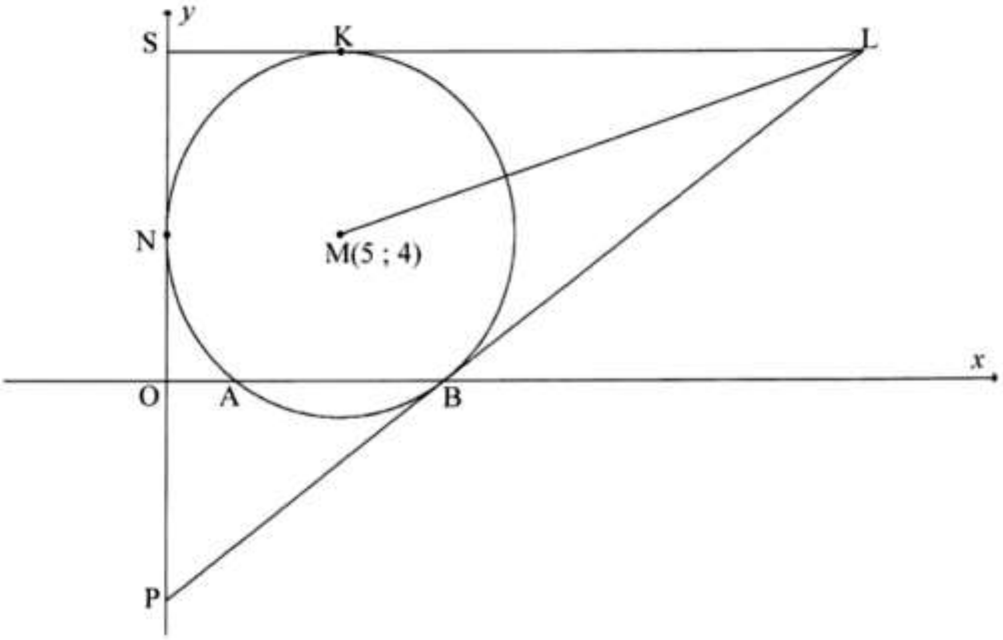
3.2 Determine the equation of the line passing through N and P in the form $y = mx + c$. (5)

3.3 Determine the length of MT. (4)

[10]

QUESTION 4

In the diagram below, a circle with centre $M(5;4)$ touches the y-axis at N and intersects the x-axis at A and B. PBL and SKL are tangents to the circle where SKL is pararell to the x-axis and P and Sare points on the y-axis. LM is drawn.



4.1. Write down the length of the radius of the circle having centre M. (1)

4.2 Write down the equation of the circle having centre M, in the form $(x-a)^2+(y-b)^2=r^2$. (1)

4.3 Calculate the coordinates of A (3)

4.4 If the coordinates of B are $(8;0)$, calculate:

4.4.1 The gradient of MB (2)

4.4.2 The equation of the tangent PB in the form $y=mx+c$. (3)

4.5 Write down the equation of tangent SKL. (2)

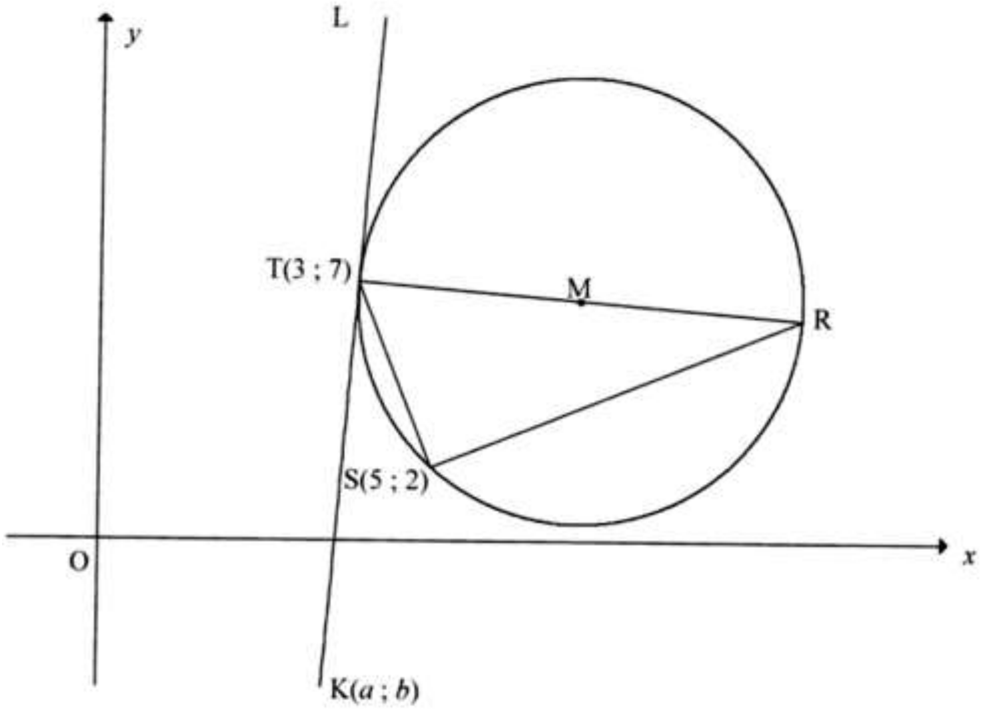
4.6 Show that L is a point$(20;9)$. (2)

4.7 Calculate the length of mL in surd form. (2)

[16]

QUESTION 5

In the diagram, M is the centre of the circle passing through $T(3; 7)$, R and $S(5; 2)$. RT is a diameter of the circle. K $(a; b)$ is a point in the 4th quadrant such that KTL is a tangent to the circle at T.



5.1 Give a reason why $T\hat{S}R = 90°$. (1)

5.2 Calculate the gradient of TS. (2)

5.3 Determine the equation of the line SR in the form $y = mx + c$. (3)

5.4 The equation of the circle above is $(x - 9^2 + (y - 6\frac{1}{2})^2 = 36\frac{1}{4}$.

5.4.1 Calculate the length of TR in surd form. (2)

5.4.2 Calculate the coordinates of R. (3)

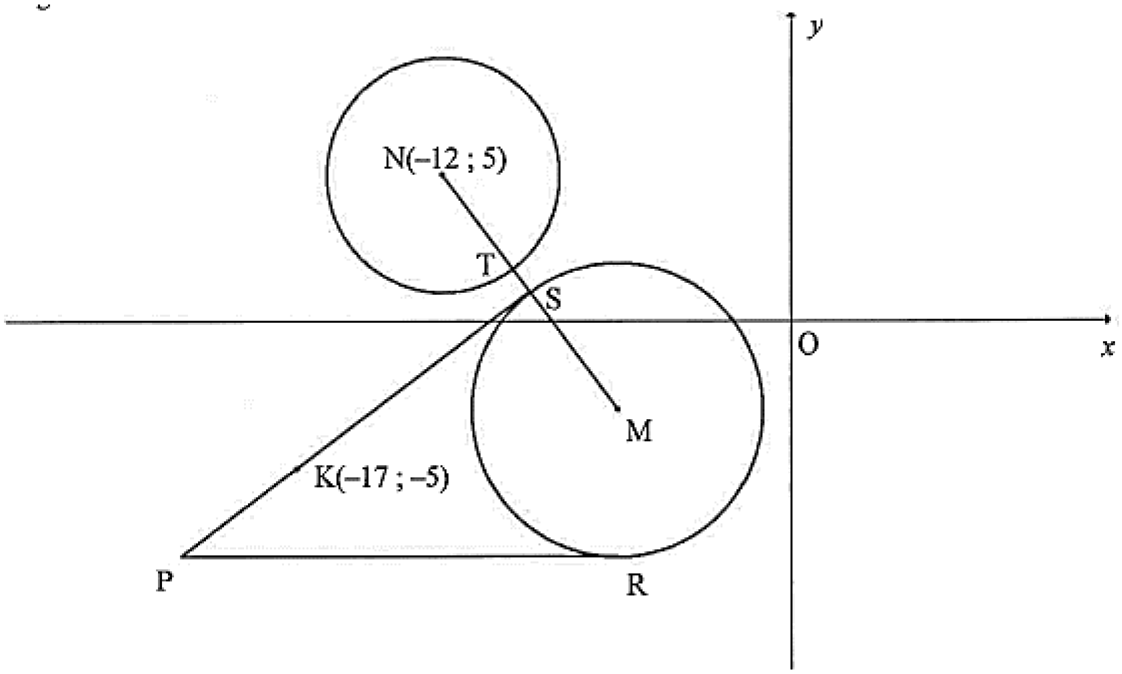
5.4.3 Calculate $sinR$. (3)

5.4.4 Show that $b = 12a - 29$. (3)

[17]

QUESTION 6

In the diagram, the equation of the circle centred at N$(-12; 5)$ is $x^2 + y^2 + 24x - 10y + 153 = 0$. The equation of the circle centred at M is $(x + 6)^2 + (y + 3)^2 = 25$. PS and PR are tangents to the circle at M at S and r respectively. PR is parallel to the x-axis. K(-17;-5) is a point on PS. The Straight line joining N and M cuts the smaller circle at T and the larger circle at S.



6.1 Write down the coordinates of M. (2)

6.2 Calculate the:

6.2.1 Length of the radius of the smaller circle. (2)

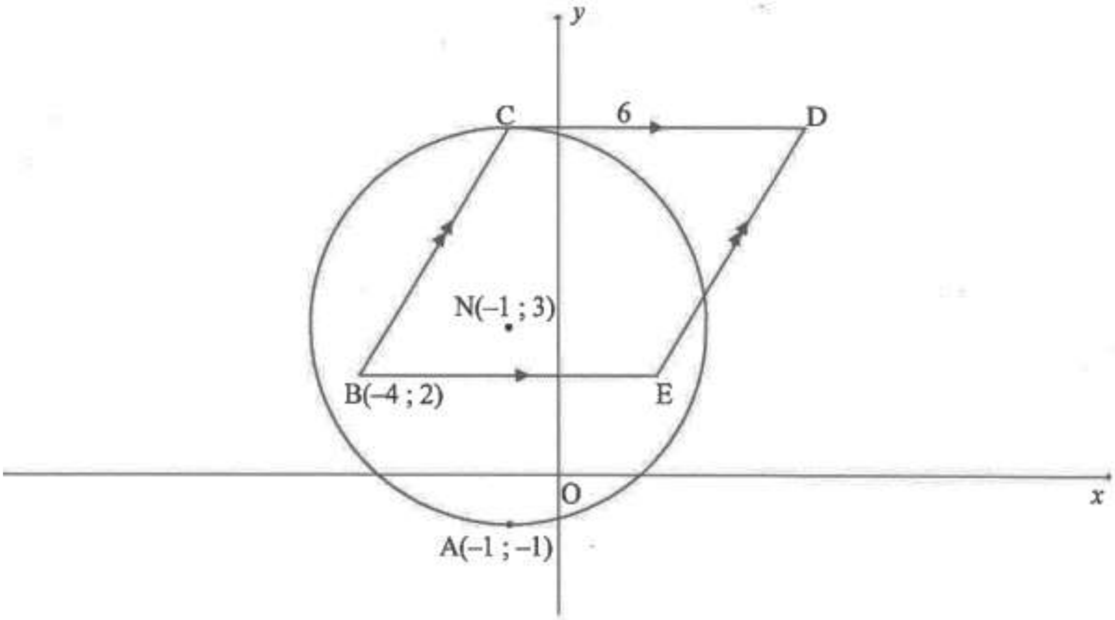
6.3 Determine the equation of the tangent:

6.3.1 PR (2)

[6]

QUESTION 7

In the diagram, the circle centred at N $(-1; 3)$ passes through A$(-1; -1)$ and C. B$(-4; 2)$,C, D and E are joined to form a parallelogram such that BE is parallel to the x-axis.CD is a tangent to the circle at C and CD = 6 units.



7.1 Write down the length of the radius of the circles. (1)

7.2 Calculate the:

7.2.1 Coordinates of C (2)

7.2.2 Coordinates of D (2)

7.3 The circle ,centred at N, is reflected about the line $y = x$. M is the centre of the new circle which is formed. The two circles intersect at A and F.

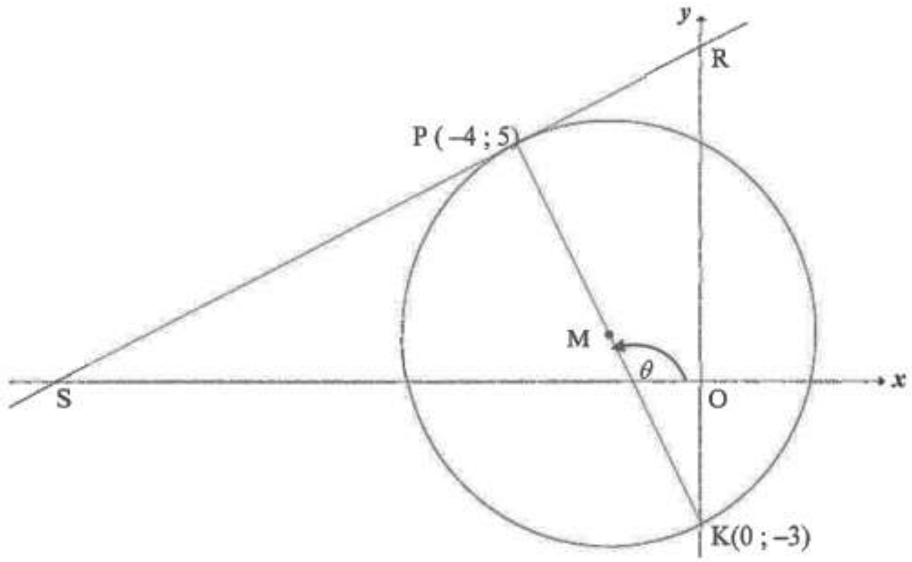
Calculate the:

7.3.1 Length of NM. (3)

[8]

QUESTION 8

In the diagram, $P(-4; 5)$ and $K(0; -3)$ are the end points of the diameter of a circle with centre M. S and R are respectively the x-axis and y-intercept of the tangent to the circle at P. θ is the inclination of PK with the positive x-axis.



8.1 Deterrmine:

8.1.1 The gradient of SR (4)

8.1.2 The equation of SR in the form $y = mx + c$ (2)

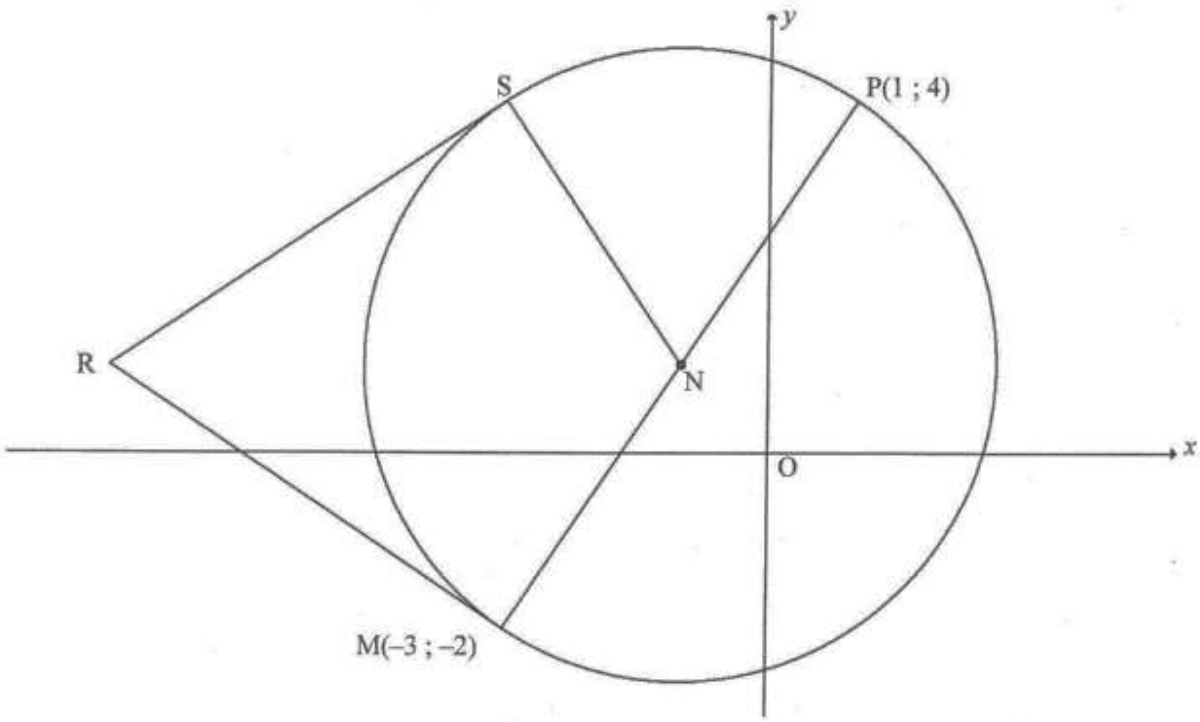
8.1.3 The equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$ (2)

8.1.4 The equation of the tangent to the circle at K in the form $y = mx + c$ (2)

[10]

QUESTION 9

In the diagram, N is the centre of the circle. $M(-3;-2)$ and $P(1;4)$ are points on the circle. MNP is the diameter of the circle. Tangents drawn to circle N from point R, outside the circle, meet the circle at S and M respectively.



9.1 Determine the coordinates of N. (3)

9.2 Determine the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$. (4)

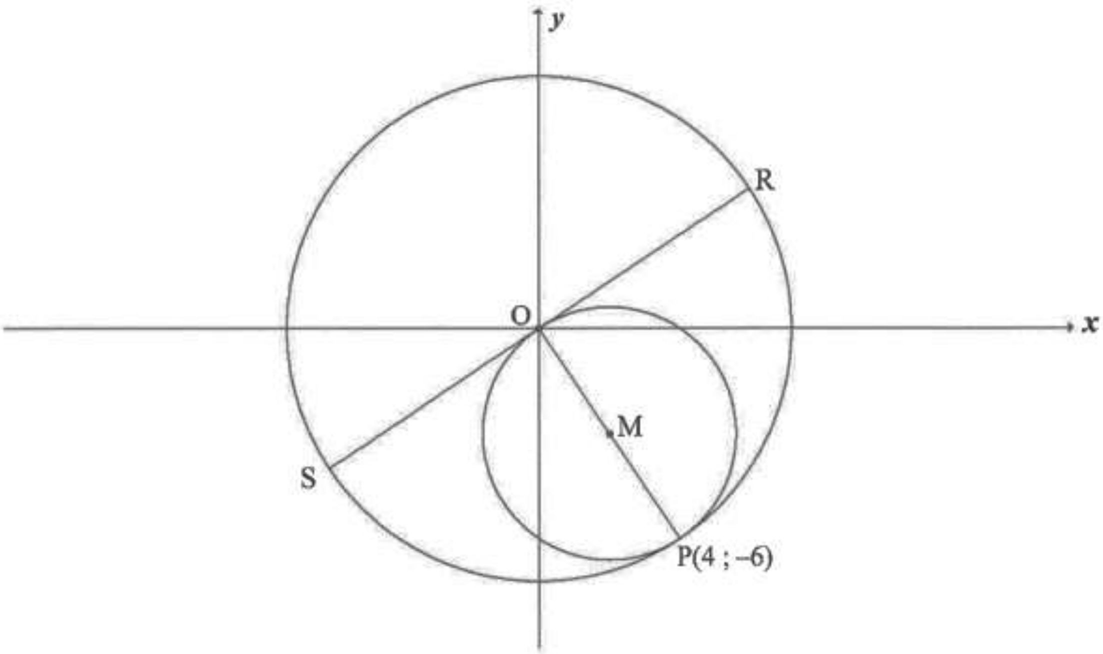
9.3 Determine the equation of the tangent RM in the form $y=mx+c$ (5)

9.4 If it is given that the line joining S to M is perpendicular to the x-axis, determine the coordinates of S. (2)

[14]

QUESTION 10

In the diagram, a circle having centre at the origin passes through $P(4;-6)$. PO is the diameter of a smaller circle having centre at M. the diameter RS of the larger circle is a tangent to the smaller circle at O.



10.1 Calculate the coordinates of M. (2)

10.2 Determine the equation of:

10.2.1 The large circle (2)

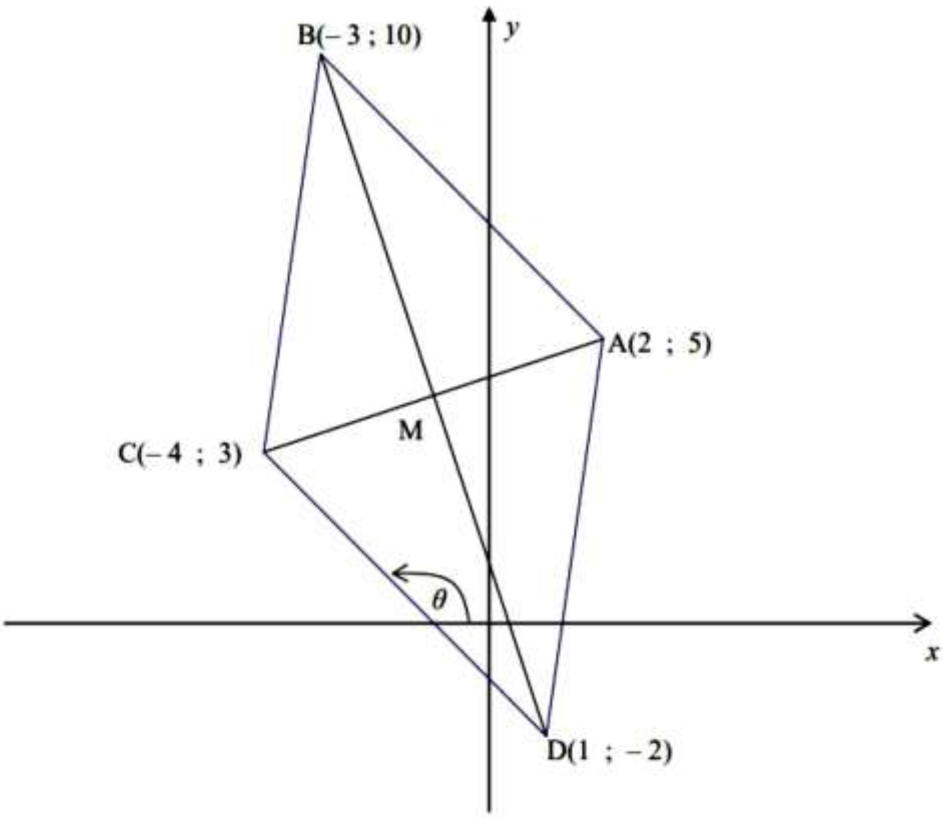
10.2.2 The equation of RS in the form $y = mx + c$ (3)

[7]

EXAM-BASED QUESTIONS OF COGNITIVE LEVEL 3 AND 4

QUESTION 1

$ABCD$ is a quadrilateral with vertices $A(2;5), B(-3;10), C(-4;3)$and $D(1;-2)$.



1.7. Calculate the length AC. (Leave the answer in simplest surd form). (2)

1.8. Determine the coordinates of M, the midpoint of AC. (2)

1.9. Show that BD and AC bisect each other at right angles at M. (5)

1.10. Calculate the area of ΔABC. (4)

1.11. Determine the equation of DC. (3)

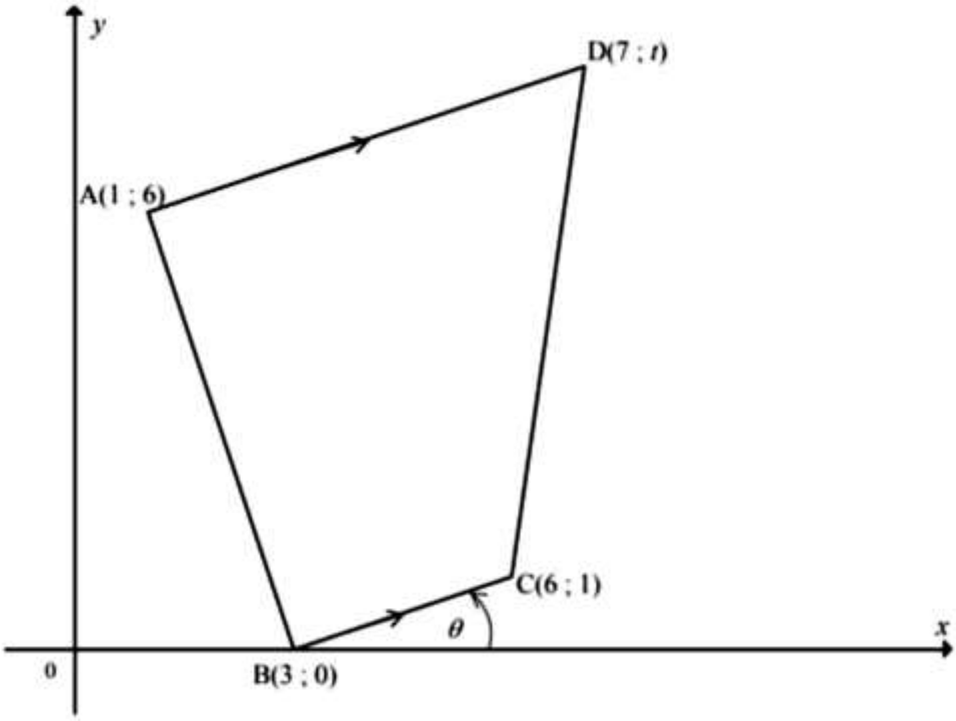
1.12. Determine θ, the angle of inclination of DC. (2)

1.13. Calculate the size of ADC (4)

[22]

QUESTION 2

$ABCD$ is a quadrilateral with vertices $A(1;6), B(3;0), C(6;1)$ and $D(7:t)$ in a Cartesian plane $AD \parallel BC$.



2.1. Calculate the gradient of BC. (2)

2.2. Determine the equation of AD in the form $y = \cdots$ (3)

2.3. Show that $t = 8$. (2)

2.4. Calculate the lengths of AD, BC and AB. (4)

2.5. Show that AB is perpendicular to BC. (3)

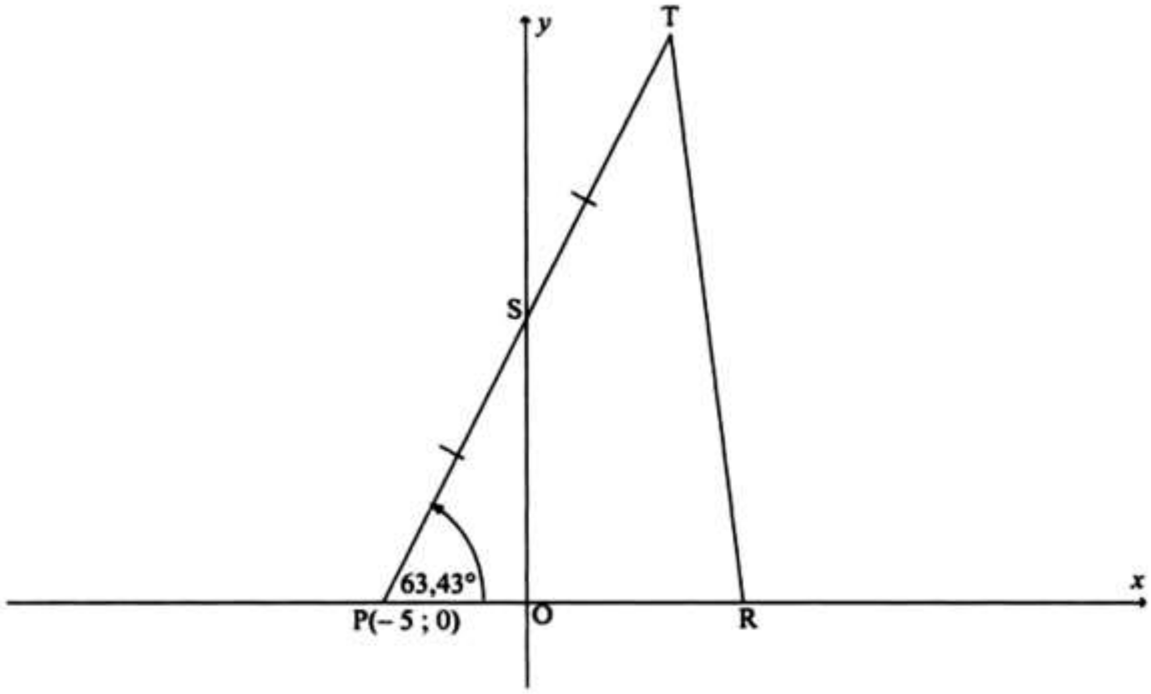
2.6. Determine θ, the angle of inclination of BC. (3)

2.7 calculate the area of abcd. (simplify your answer) (4)

[21]

QUESTION 3

In the diagram below, P is a point $(-5;0)$. The inclination of line PT is $63{,}43°$. S is the midpoint and the y-intercept of PT. R is a point on the x-axis such that $PO:OR = 2:3$.



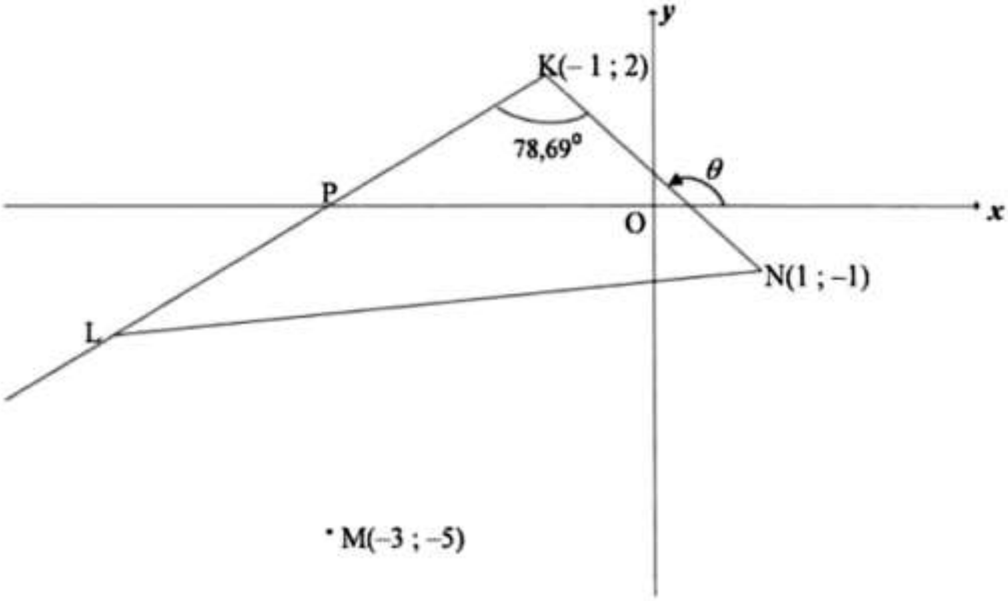
3.1. Determine:

3.1.1.	The gradient of PT, correct to the nearest integer value.	(2)
3.1.2.	The equation of PT in the form $y = mx + c$.	(2)
3.1.3.	The distance PS in surd form.	(3)
3.1.4.	The coordinates of T.	(2)
3.1.5.	Determine the coordinates of R.	(2)
3.1.6	Calculate the area of ΔPTR.	(4)

[15]

QUESTION 4

In the diagram, $K(-1;2), L$ and $N(1;-1)$ are vertices of ΔKLN such that $L\hat{K}N = 78{,}69°$. KL intersects the x-axis at P. KL is produced. The inclination of KN is θ. The coordinates of M are (-3;-5).



4.1. Calculate:

4.1.1. The gradient of KN. (2)

4.1.2. The size of θ, the inclination of KN. (2)

4.2. Show that the gradient of KL is 1. (2)

4.3. Determine the equation of the straight line KL in the form $y = mx + c$. (2)

4.4. Calculate the length of KN. (2)

4.5 It is further given that KN =LM.

4.5.1. Calculate the possible coordinates of L. (5)

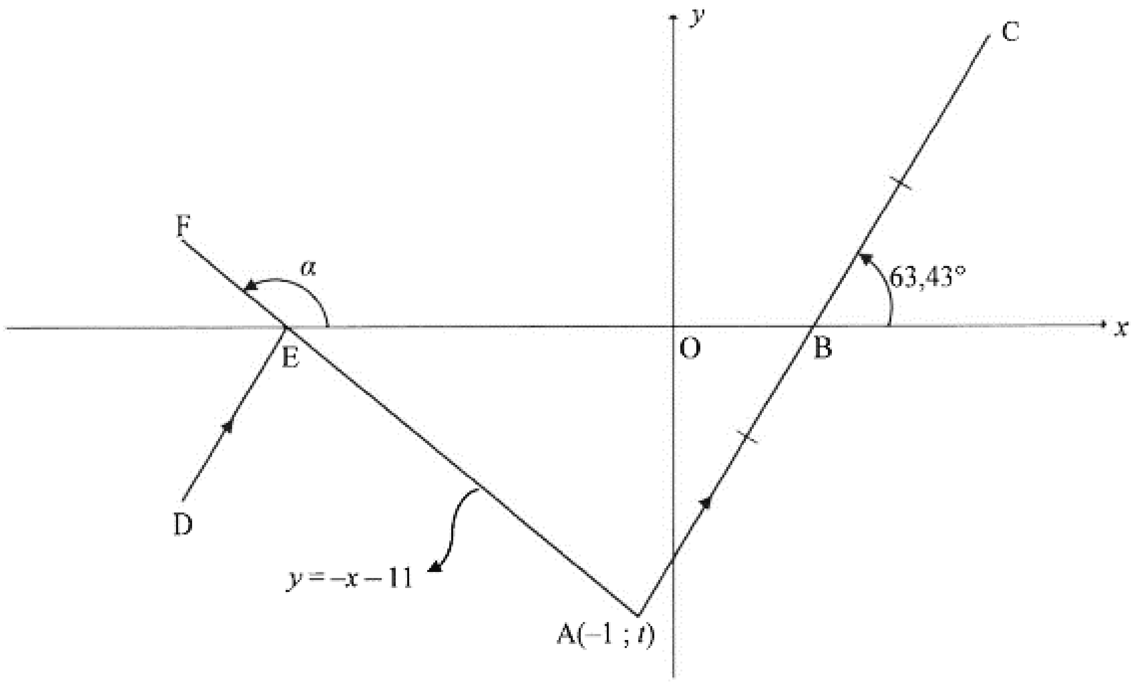
4.5.2. Determine the coordinates of L if it is given that KLMN is a parallelogram. (3)

4.6. T is a point on KL produced. TM is drawn such that TM = LM. Calculate the area of ΔKTN. (4)

[22]

QUESTION 5

In the diagram, the equation of line AF is $y = -x - 11$. B, a point on the x-axis is the midpoint of the straight line joining $A(-1; t)$ and C. The angles of inclination of AF and AC are α and $63{,}43°$ respectively. AF cuts the x-axis in E. d is a point such that DE∥AC.



5.1 Calculate the:

5.1.1 Value of t (2)

5.1.2 Size of α (2)

5.1.3 Gradient of AC, to the nearest whole number (2)

5.2 Determine the equation of AC in the form $y = mx + k$ (2)

5.3 Calculate the:

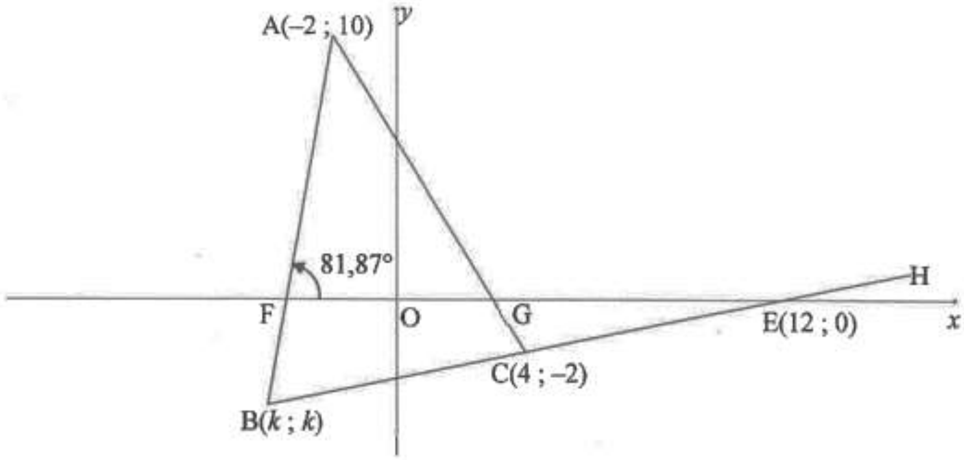
5.3.1 Coordinates of C (3)

5.3.2 Size of $F\hat{E}D$ (3)

[11]

QUESTION 6

In the diagram, $A(-2;10), B(k;k)$ and $C(4;;-2)$ are the vertices of ΔABC. Line BC is produced to H and cuts the x-axis at $E(12;0)$. AB and AC intersects the x-axis at F and G respectively. The angle of inclination of line AB is81,87°.



6.1. Calculate the gradient of:

6.1.1. BE (2)

6.1.2. AB (2)

6.2. Determine the equation of BE in the form $y = mx + c$ (2)

6.3. Calculate:

6.3.1. Coordinates of B, where $k < 0$ (2)

6.3.2 The size of A (4)

6.3.2. Coordinates of the point of intersection of the diagonals of parallelogram ACES, where S is a point in the first quadrant. (2)

6.4. Another point $T(p;p)$, where $p > 0$, is plotted such that $ET = BE = 4\sqrt{17}$ units.

6.4.1. Determine the equation of the.

a) Circle with centre at E and passing through B and T in the form

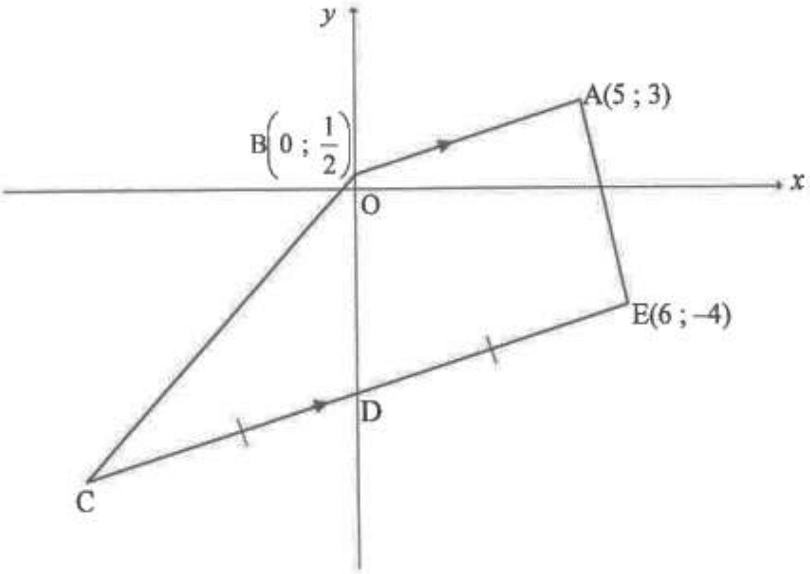
$(x-a)^2 + (y-b)^2 = r^2$ (2)

b) Tangents to the circle at point $B(k;k)$ (3)

[24]

QUESTION 7

In the diagram, $A(5;3), B\left(0;\frac{1}{2}\right), C$ and $E(6;-4)$ are the vertices of a trapezium having $BA \parallel CE$. D is the y-intercept of CE and $CD = DE$.



7.1 Calculate the gradient of AB (2)

7.2 Determine the equation of CE in the form $y = mx = c$ (3)

7.3 Calculate the:

7.3.1 Coordinates of C (3)

7.3.2 Area of quadrilateral ABCD (4)

7.4 If point K is the reflection of E in the y-axis:

7.4.1 Write down the coordinates of K (2)

7.4.2 Calculate the:

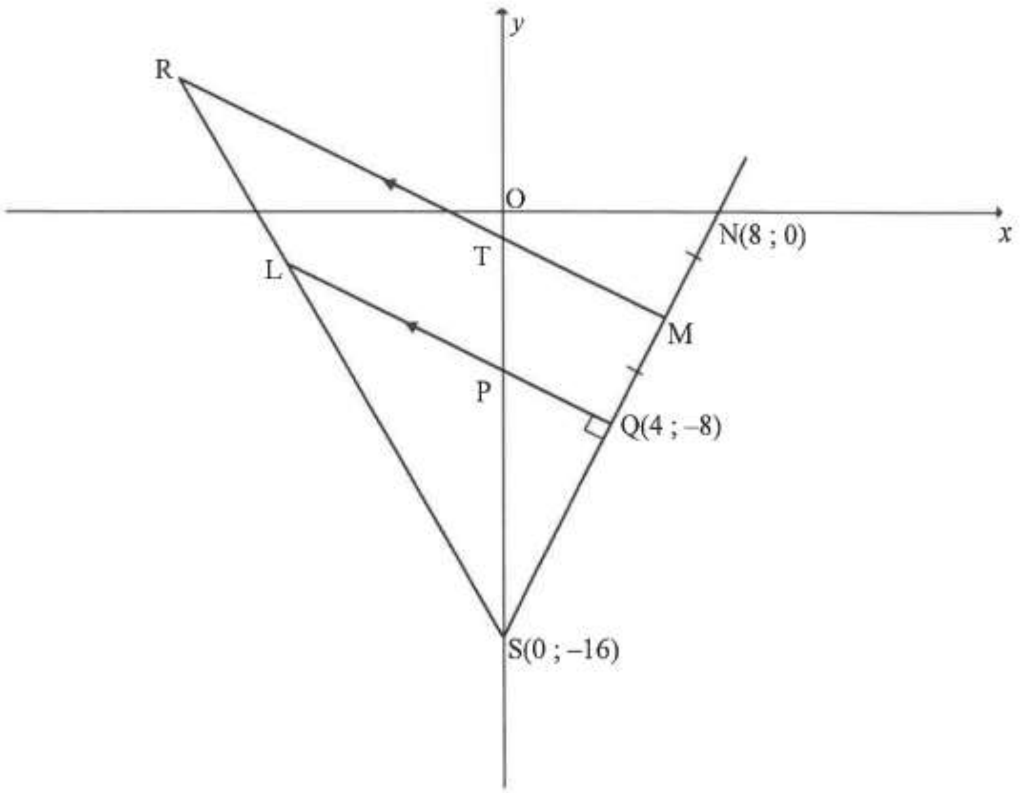
(a) Perimeter of ΔKEC (4)

(b) Size of $K\hat{C}E$ (3)

[14]

QUESTION 8

In the diagram, $S(0;-16)$, L and $Q(4;-8)$ are the vertices of ΔSLQ having LQ perpendicular to SQ. SL and SQ are produced to points R and M respectively such that RM ∥ LQ. SM produced cuts the x-axis at N $(8;0)$. QM = MN. T and P are the y-intercepts of RM and LQ respectively.



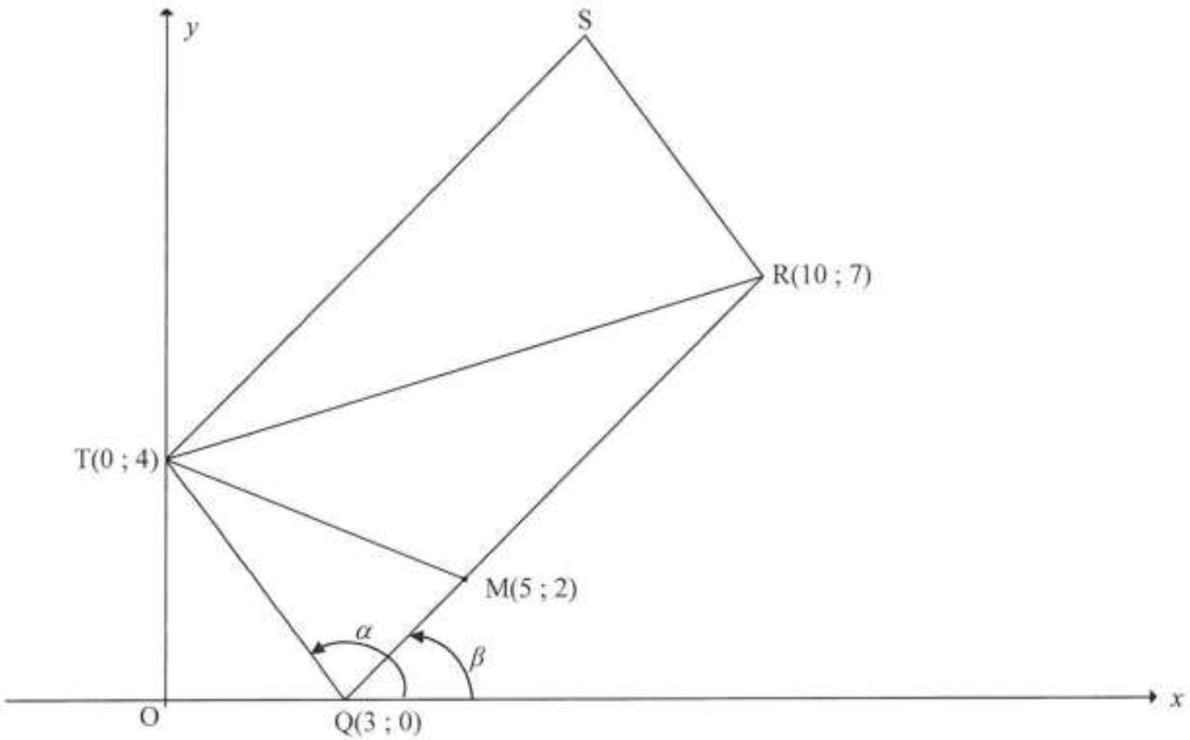
8.1	Calculate the coordinates of M.	(2)
8.2	Calculate the gradient of NS.	(2)
8.3	Show that the equation of line LQ is $y = \frac{-1}{2}x - 6$.	(3)
8.4	Determine the equation of a circle having a centre at O, the origin, and also passing through S	(2)
8.5	Calculate the coordinates of T.	(3)
8.6	Determine $\frac{\text{LS}}{\text{RS}}$	(3)
8.7	Calculate the area of PTMQ.	(4)

[19]

QUESTION 9

In the diagram, $Q(3;0), R(10;7), S$ and $T(0;4)$ are the vertices of parallelogram QRST.

From T a straight line is drawn to meet QR at $M(5;2)$. The angles of inclination of TQ and RQ are α and β respectively.



9.1. Calculate the gradient of TQ. (1)

9.2. Calculate the length of RQ. Leave your answer in surd form. (2)

9.3. $F(k;-8)$ is a point in the Cartesian plane such that T, Q and F are collinear. Calculate the value of k. (4)

9.4 Calculate the coordinates of S (4)

9.5 Calculate the size of $T\hat{S}R$. (6)

9.6 Calculate, in the simplest form, the ratio of:

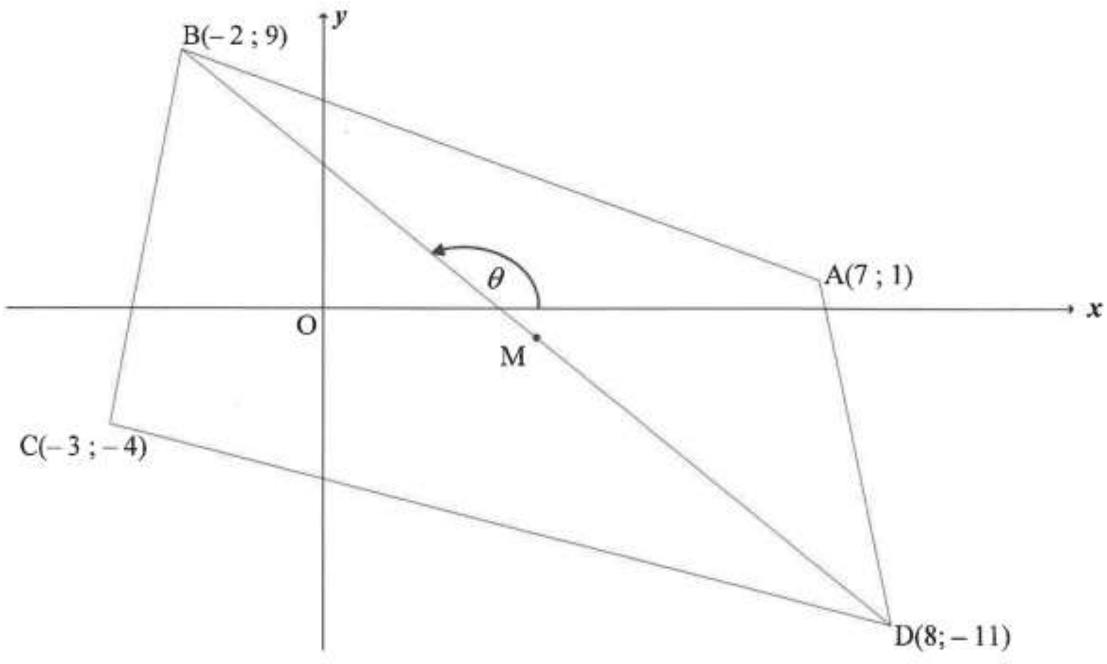
9.6.1 $\dfrac{\text{MQ}}{\text{RQ}}$ (3)

9.6.2 $\dfrac{\text{Area of }\Delta\text{TQM}}{\text{Area of parallelogram RQTS}}$ (3)

[23]

QUESTION 10

In the diagram, $ABCD$ is a parallelogram having vertices $A(7;1), B(-2;9), C(-3:-4)$ and $D(8;-11)$. M is the midpoint of BD.



10.1. Calculate the gradient of AC. (2)

10.2. Determine:

10.2.1. The equations of AC in the form $y = mx + c$ (2)

10.2.2. Whether M lies on AC (4)

10.3. Prove that $BD \perp AC$. (3)

10.4. Calculate:

10.4.1. θ, the inclination of BD. (2)

10.4.2 the size of $C\hat{B}D$ (3)

10.4.3. The length of AC. (2)

10.4.4 The area of ABCD (5)

[23]

QUESTION 1

1.1 Determine the centre and radius of the circle with the equation
$x^2 + y^2 + 8x + 4y - 38 = 0$ (4)

1.2 A second circle has the equation $(x-4)^2 + (y-6)^2 = 26$. Calculate the distance between the centres of the two circles. (2)

1.3 Hence, show that the circles described in 1.1 and 1.2 intersect each other. (3)

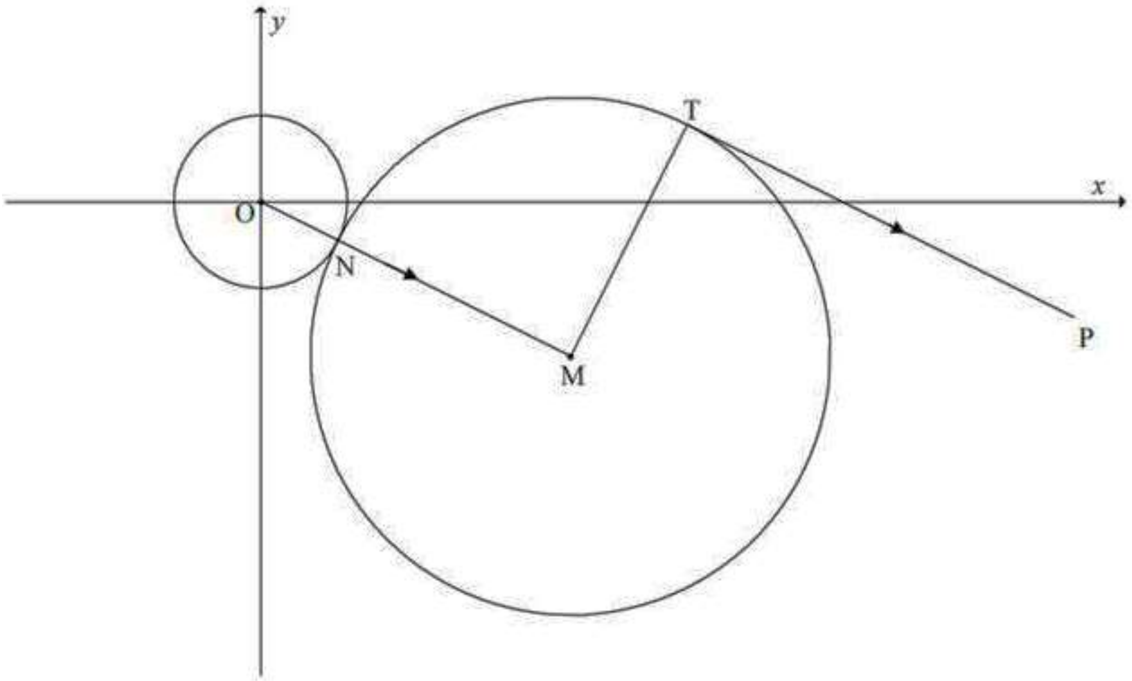
1.4 Show that the two circles intersect along the line $y = -x + 4$ (4)

[13]

QUESTION 2

In the diagram below, the equation of the circle with centre M is $(x-8)^2 + (y+4)^2 = 45$.

PT is a tangent to this circle at T and PT is parallel to OM. Another circle having centre O, touches the circles having centre M at N.



2.1 Write down the coordinates of M. (1)

2.2 Calculate the length of OM. Leave your answer in simplest form. (2)

2.3 Calculate the length of ON. Leave your answer in simplest form. (3)

2.4 Calculate the size of $O\hat{M}T$. (2)

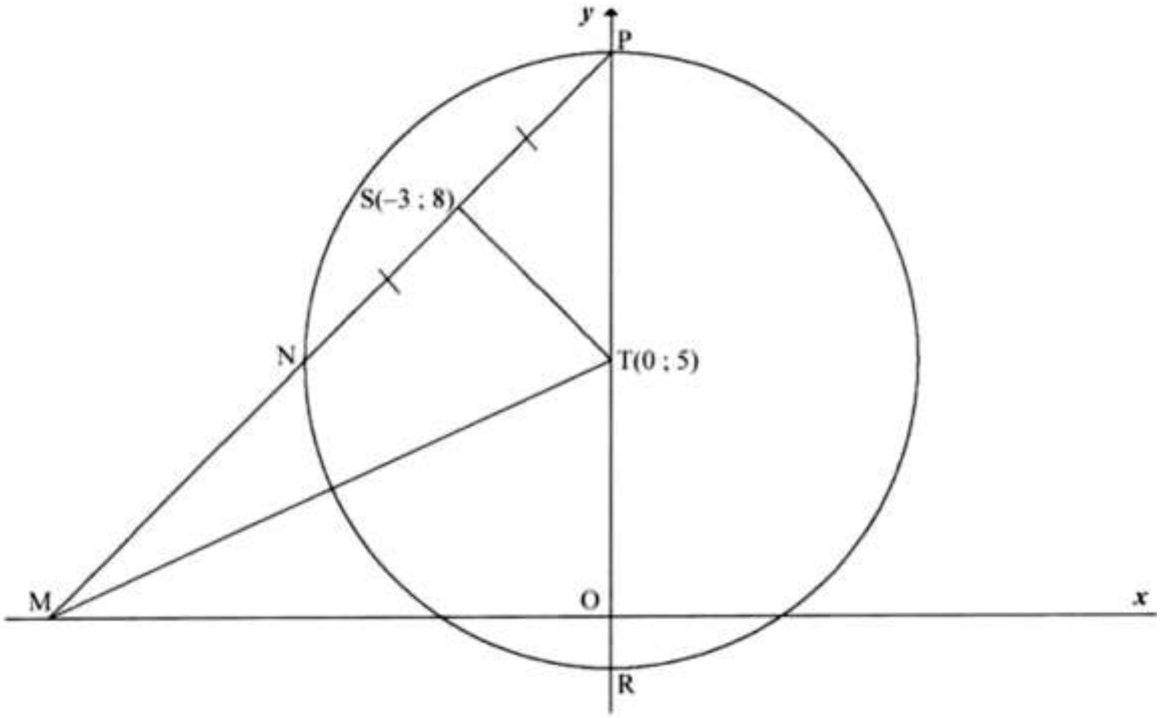
2.5 Determine the equation of MT in the form $y = mx + c$. (5)

[13]

QUESTION 3

In the diagram, the circle, having centre $T(0; 5)$, cuts the y-axis at P qnd R. The line through P and $S(-3; 8)$ intersects the circle at N and the x-axis at M. NS =PS.

MT is drawn.



3.1 Give a reason why TS ⊥ NP. (1)

3.2 Determine the equation of the line passing through N and P in the form $y = mx + c$. (5)

3.3 Determine the length of MT. (4)

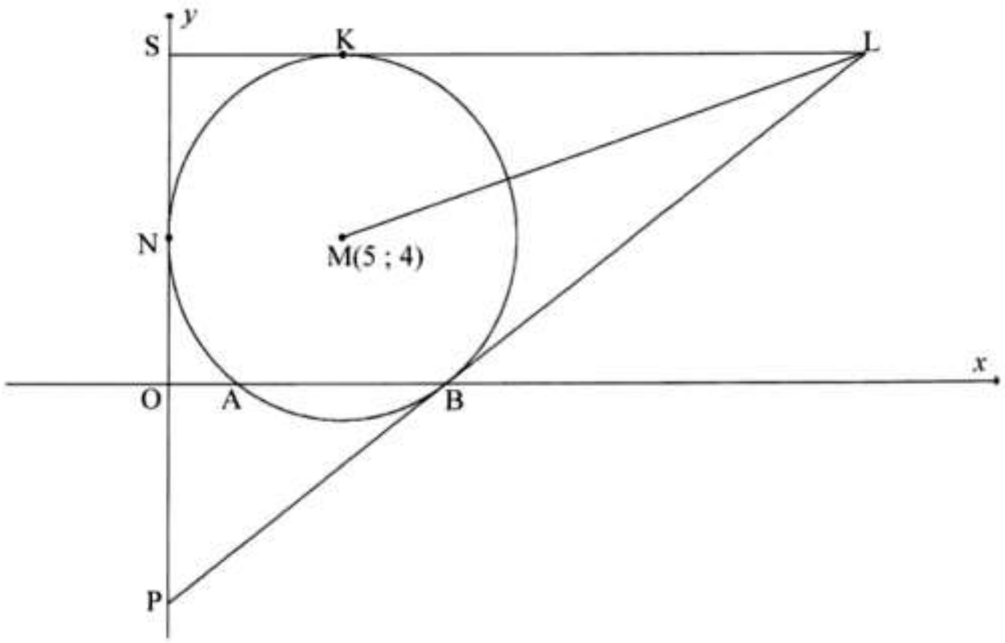
3.4 determine the equation of the tangents to the circle that are parallel to the $x - axis$. (4)

3.5 Another circle is drawn through the points S, T and M, determine the equation, with the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$. (5)

[19

QUESTION 4

In the diagram below, a circle with centre $M(5; 4)$ touches the y-axis at N and intersects the x-axis at A and B. PBL and SKL are tangents to the circle where SKL is pararell to the x-axis and P and Sare points on the y-axis. LM is drawn.



4.1. Write down the length of the radius of the circle having centre M. (1)

4.2 Write down the equation of the circle having centre M, in the form

$(x-a)^2+(y-b)^2=r^2$. (1)

4.3 Calculate the coordinates of A (3)

4.4 If the coordinates of B are $(8; 0)$, calculate:

4.4.1 The gradient of MB (2)

4.4.2 The equation of the tangent PB in the form $y = mx + c$. (3)

4.5 Write down the equation of tangent SKL. (2)

4.6 Show that L is a point$(20; 9)$. (2)

4.7 Calculate the length of mL in surd form. (2)

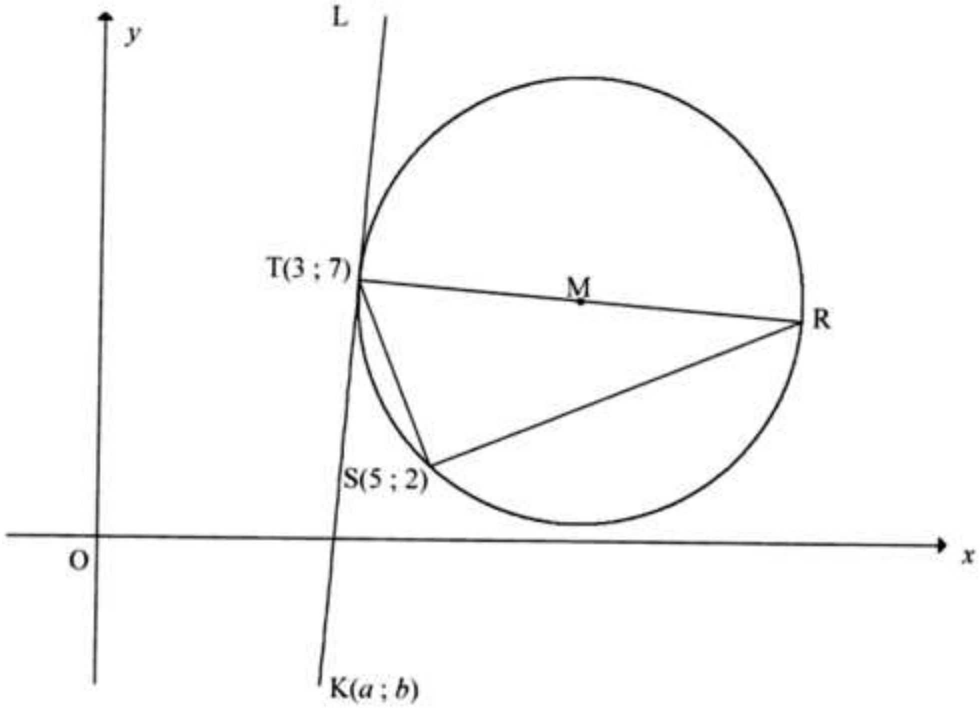
4.8 determine the equation of the circle passing through points K,L and M in the form

$(x-p)^2 + (y-p)^2 = c^2$ (5)

[21]

QUESTION 5

In the diagram, M is the centre of the circle passing through $\mathrm{T}(3;7)$, R and $\mathrm{S}(5;2)$. RT is a diameter of the circle. K $(a;b)$ is a point in the 4th quadrant such that KTL is a tangent to the circle at T.



5.1 Give a reason why $T\hat{S}$R = 90°. (1)

5.2 Calculate the gradient of TS. (2)

5.3 Determine the equation of the line SR in the form $y = mx + c$. (3)

5.4 The equation of the circle above is $(x-9^2 + (y-6\frac{1}{2})^2 = 36\frac{1}{4}$.

5.4.1 Calculate the length of TR in surd form. (2)

5.4.2 Calculate the coordinates of R. (3)

5.4.3 Calculate $sinR$. (3)

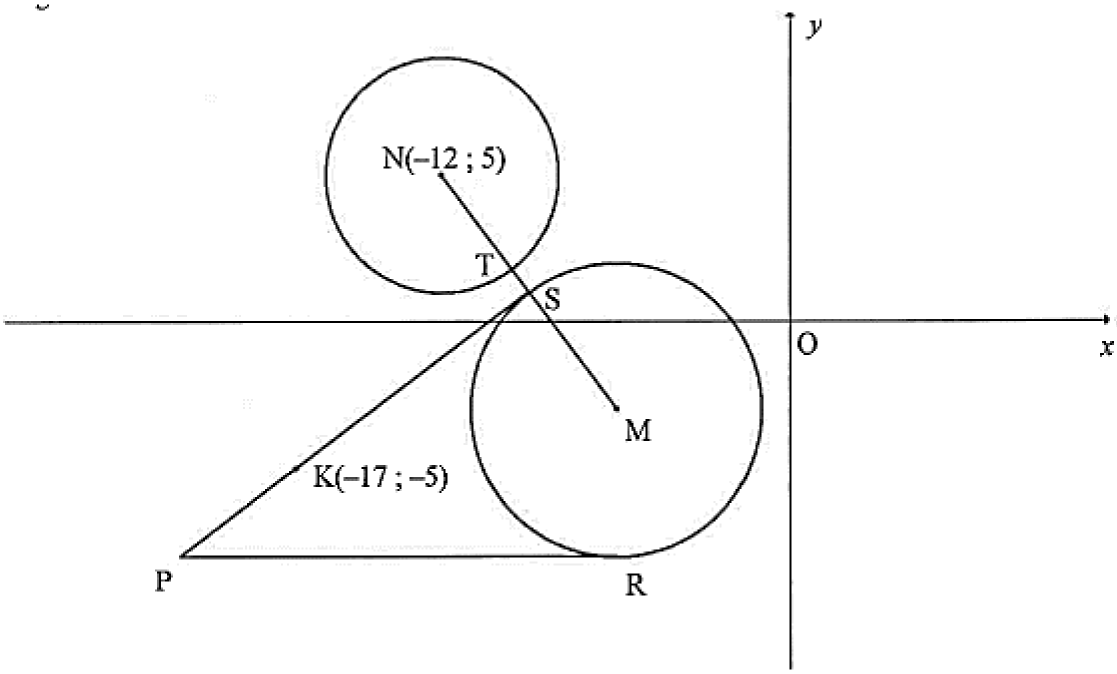
5.4.4 Show that $b = 12a - 29$. (3)

5.4.5 If TK = TK, calculate the coordinates of K. (6)

[23]

QUESTION 6

In the diagram, the equation of the circle centred at N$(-12;5)$ is $x^2+y^2+24x-10y+153=0$. The equation of the circle centred at M is $(x+6)^2+(y+3)^2=25$. PS and PR are tangents to the circle at M at S and r respectively. PR is parallel to the x-axis. K(-17;-5) is a point on PS. The Straight line joining N and M cuts the smaller circle at T and the larger circle at S.



6.1 Write down the coordinates of M. (2)

6.2 Calculate the:

6.2.1 Length of the radius of the smaller circle. (2)

6.2.2 Length of TS (4)

6.3 Determine the equation of the tangent:

6.3.1 PR (2)

6.3.2 PS, in the form $y=mx+c$ (5)

6.4 Quadrilateral PSMR is drawn, calculate the:

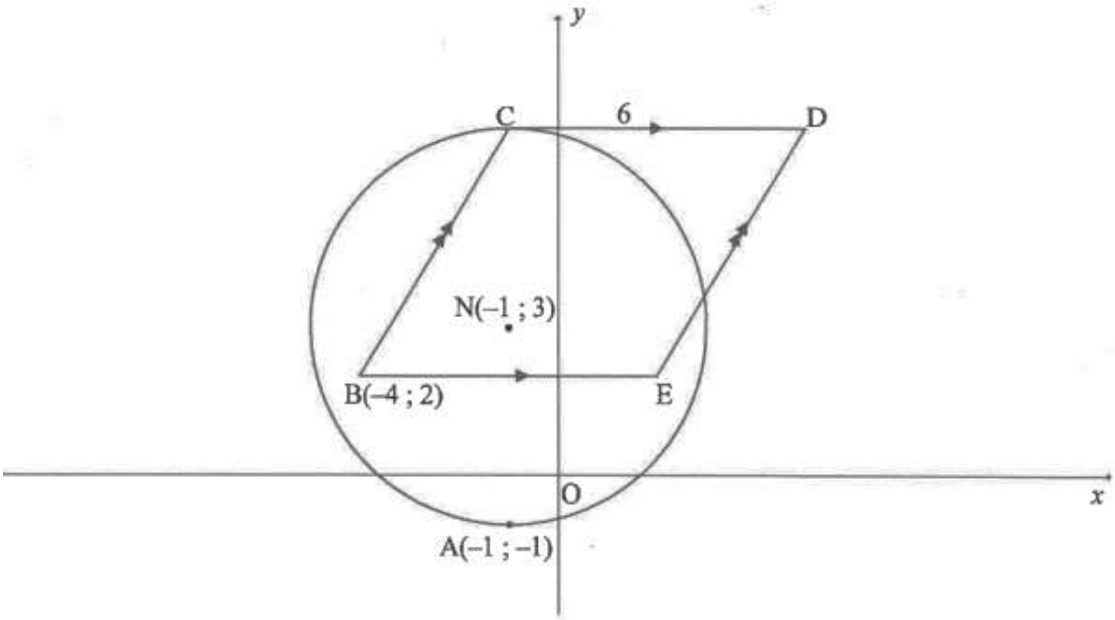
6.4.1 Perimeter of PSMR (5)

6.4.2 Ratio of $\frac{\text{Area of }\Delta\text{NPS}}{\text{area of quadrilateral PSMR}}$ (2)

[22]

QUESTION 7

In the diagram, the circle centred at N $(-1; 3)$ passes through A$(-1; -1)$ and C. B$(-4; 2)$,C, D and E are joined to form a parallelogram such that BE is parallel to the x-axis.CD is a tangent to the circle at C and CD = 6 units.



7.1 Write down the length of the radius of the circles. (1)

7.2 Calculate the:

7.2.1 Coordinates of C (2)

7.2.2 Coordinates of D (2)

7.2.3 Area of ΔBCD (3)

7.3 The circle ,centred at N, is reflected about the line $y = x$. M is the centre of the new circle which is formed. The two circles intersect at A and F.

Calculate the:

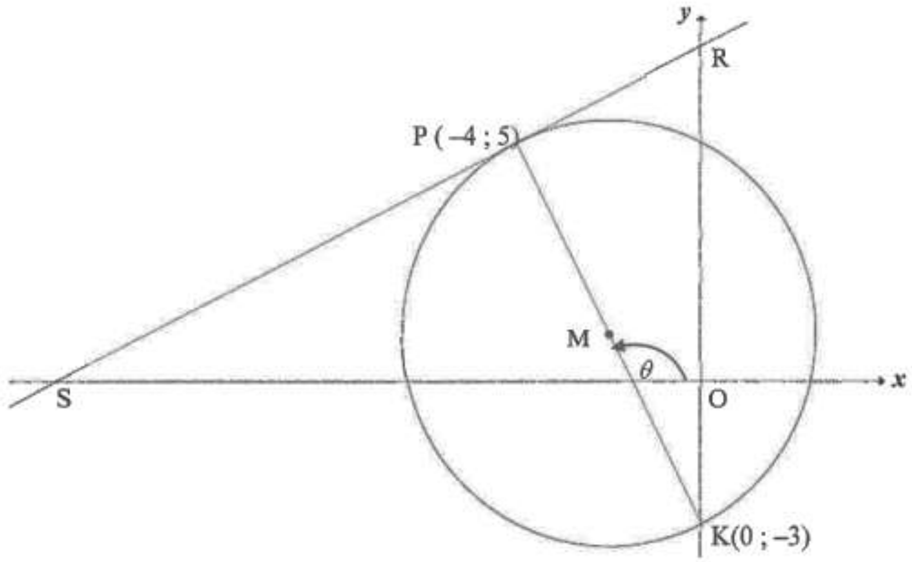
7.3.1 Length of NM. (3)

7.3.2 Midpoint of AF (4)

[15]

QUESTION 8

In the diagram, $P(-4; 5)$ and $K(0; -3)$ are the end points of the diameter of a circle with centre M. S and R are respectively the x-axis and y-intercept of the tangent to the circle at P. θ is the inclination of PK with the positive x-axis.



Deterrmine:

8.1.1 The gradient of SR (4)

8.1.2 The equation of SR in the form $y = mx + c$ (2)

8.1.3 The equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$ (2)

8.1.4 The size of $P\hat{K}R$ (3)

8.1.5 The equation of the tangent to the circle at K in the form $y = mx + c$ (2)

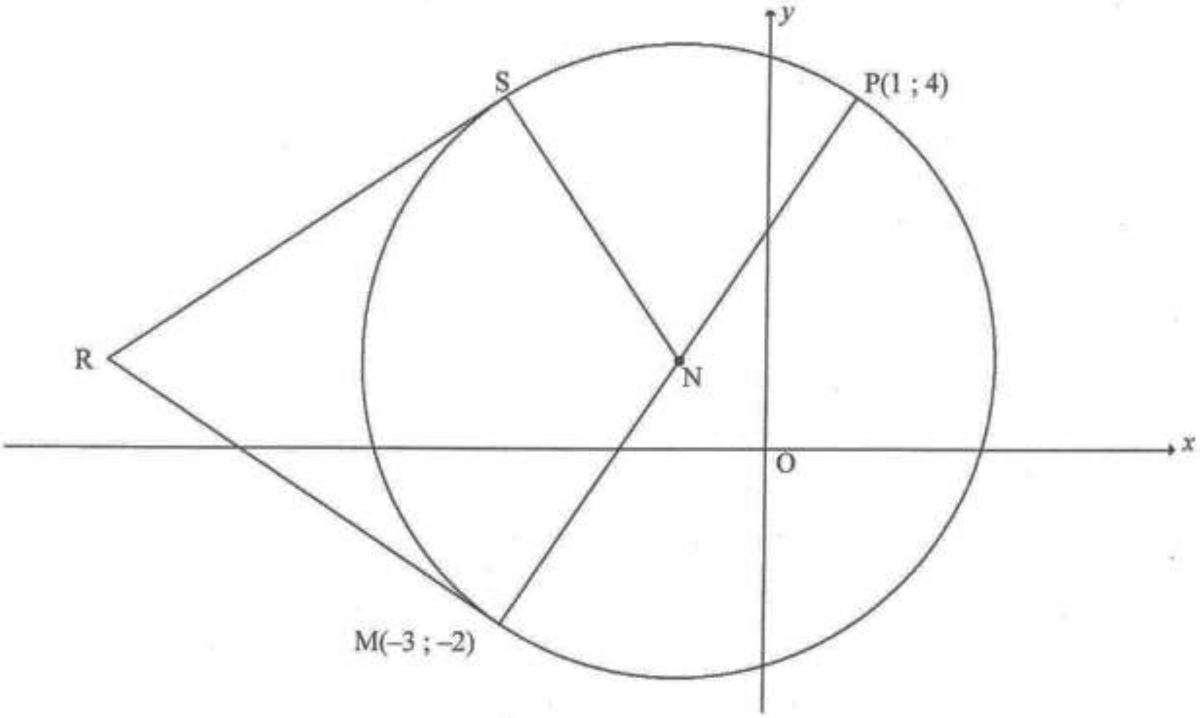
8.2 determine the values of t such that the line $y = \frac{1}{2}x + t$ cuts the circle at different points (3)

8.3 Calculate the area of ΔSMK (5)

[23]

QUESTION 9

In the diagram, N is the centre of the circle. $M(-3;-2)$ and $P(1;4)$ are points on the circle. MNP is the diameter of the circle. Tangents drawn to circle N from point R, outside the circle, meet the circle at S and M respectively.



9.1 Determine the coordinates of N. (3)

9.2 Determine the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$. (4)

9.3 Determine the equation of the tangent RM in the form $y=mx+c$ (5)

9.4 If it is given that the line joining S to M is perpendicular to the x-axis, determine the coordinates of S. (2)

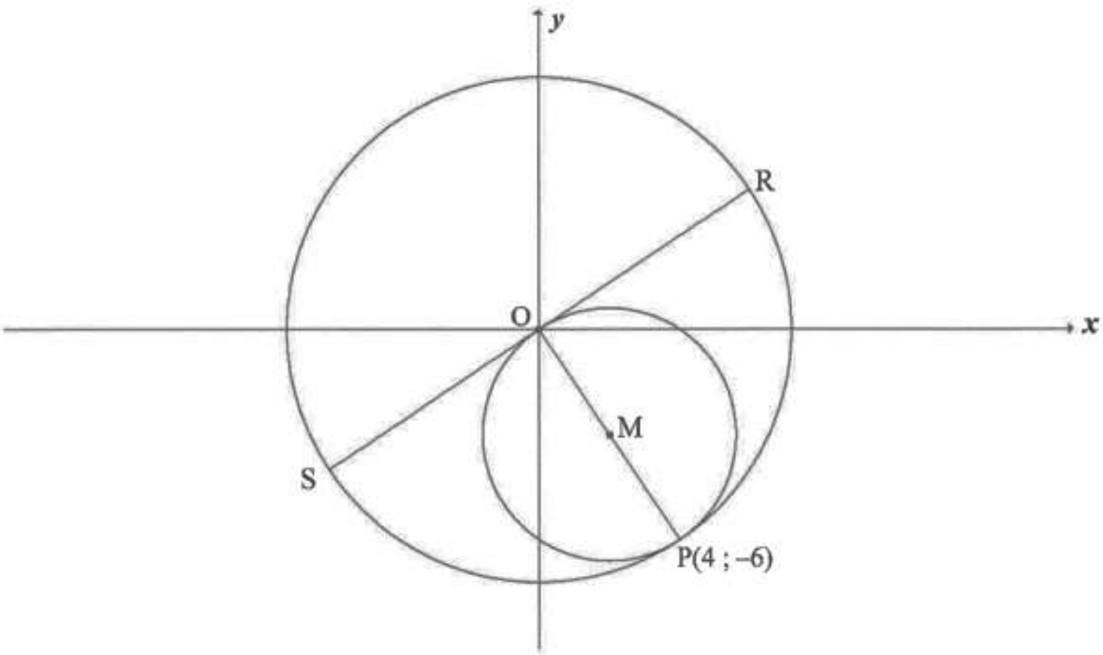
9.5 determine the coordinates of R, the common external point from which both tangents to the circle are drawn. (4)

9.6 Calculate the area of RSNM (4)

[22]

QUESTION 10

In the diagram, a circle having centre at the origin passes through $P(4;-6)$. PO is the diameter of a smaller circle having centre at M. the diameter RS of the larger circle is a tangent to the smaller circle at O.



10.1 Calculate the coordinates of M. (2)

10.2 Determine the equation of:

10.2.1 The large circle (2)

10.2.2 The equation of RS in the form $y = mx + c$ (3)

10.3 Determine the length of chord NR, where N is the reflection of R in the $y - axis$ (4)

10.4 the circle with centre at M is reflected about the $x - axis$ to for another circle centered at K. Calculate the length of the common chord of these two circles. (3)

[17]

PART C

BIBLIOGRAPHY

The following sources were consulted for the purpose of this booklet.

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Mathematics: analytical geometry: grade 12, page 1 – 2. Basic education.

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Mathematics: Examination guidelines, page 8. Basic education.

TRIGONOMETRY

Table of Contents

Trigonometry is a branch of mathematics that is more concerned about the relationships between the sides and angles of triangles.

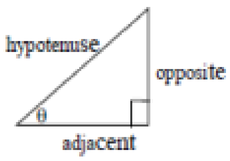
1. Definition of trig ratios:

- In a right-angled Δ

$\text{Sin}\,\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\text{Cos}\,\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\text{Tan}\,\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

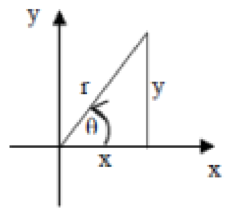


- On a Cartesian Plane

$\text{Sin}\,\theta = \dfrac{y}{r}$

$\text{Cos}\,\theta = \dfrac{x}{r}$

$\text{Tan}\,\theta = \dfrac{y}{x}$



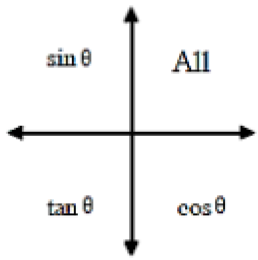
In a right angled triangle: $\sin\theta = \dfrac{opposite}{hypotenuse}$; $\cos\theta = \dfrac{adjacent}{hypotenuse}$ and $\tan\theta = \dfrac{opposite}{adjacent}$

2. Theorem of Pythagoras:

- ✓ It's given by the equation : $x^2 + y^2 = r^2$
- ✓ The Pythagoras theorem can be used anytime if we have a right angled triangle with two sides and want to find a third side.
- ✓ r is always a longest positive side and opposite to 90°

3. CAST Rule:

All trig ratios are positive in the 1st quadrant. **All**
Only $\sin\theta$ is positive in the 2nd quadrant. **Students**
Only $\tan\theta$ is positive in the 3rd quadrant. **Take**
Only $\cos\theta$ is positive in the 4th quadrant. **Care**



4. Reduction formula:

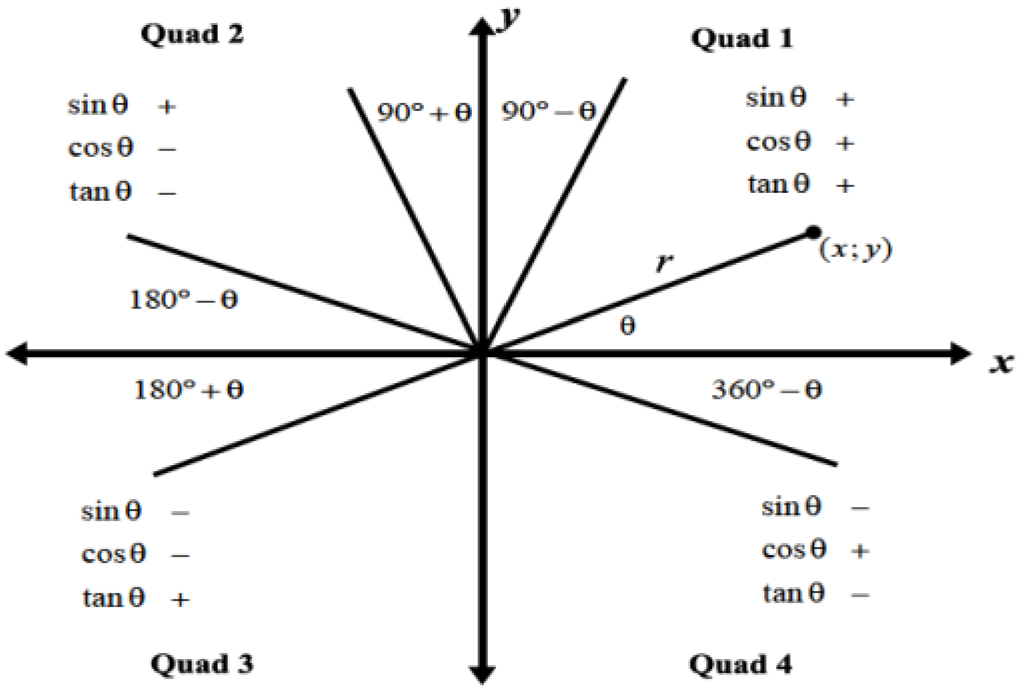
Note that :

- ➢ All positive angles are measured from positive x-axis in anti-clockwise direction.
- ➢ All negative angles are measured from positive x-axis in clockwise direction.

NOTE

- MOVING FORWARD YOU MUST ADD
- MOVING BACK YOU MUST SUBTRACT

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$



2^{nd} **Quadrant**	3^{rd} **Quadrant**	4^{th} **Quadrant**
$\sin(180^0 - \theta) = \sin\theta$	$\sin(180^0 + \theta) = -\sin\theta$	$\sin(360^0 - \theta) = -\sin\theta$
$\cos(180^0 - \theta) = -\cos\theta$	$\cos(180^0 + \theta) = -\cos\theta$	$\cos(360^0 - \theta) = \cos\theta$
$\tan(180^0 - \theta) = -\tan\theta$	$\tan(180^0 + \theta) = \tan\theta$	$\tan(360^0 - \theta) = -\tan\theta$

Tips on simplifying trigonometric expressions and equations using reduction formulae:

- ✓ Check in which quadrant does an angle belong.
- ✓ When simplifying reduction formula work within brackets.
- ✓ In that quadrant, is a function concerned negative or positive? If negative then write it down as E.g. ($-\sin\theta$), and then if positive write it as E.g. ($\sin\theta$).

- ✓ Do you change the function or not? (Functions change to their co- functions when we have ($90^0 \pm \theta$)
- ✓ For all angles that are greater than 360^0, subtract 360^0 until they are less than 360^0 and reduce accordingly.
- ✓ For trig ratios with exponent 2, use square brackets and put the exponent outside the bracket and then simplify inside the bracket. E.g. $\sin^2(180^0 + x) = [\sin(180^0 + x)]^2$, therefore, trig ratios with exponent 2 will ALWAYS give a positive answer. E.g $(-\sin x)^2 = (-\sin x)(-\sin x) = sin^2 x$

5. Co – functions:

- ✓ Ratios change to their co- functions when we have $90^0 \pm \theta$.
- ✓ The co- function of Sin is Cos and the co – function of Cos is Sin.
- ✓ Co – functions are simplified in a similar method of reduction formula, the difference is that in here ratio changes to the co- function.

1st Quadrant	2nd Quadrant
$\sin(90^0 - \theta) = \cos\theta$	$\sin(90^0 + \theta) = \cos\theta$
$\cos(90^0 - \theta) = \sin\theta$	$\cos(90^0 + \theta) = -\sin\theta$

6. Complementary angles:

- ✓ Complementary angles add up to 90^0.
- ✓ Co- ratios of complementary angles are equal.
- ✓ i.e. $\sin(40^0) = \cos(50^0)$

7. Negative angles

- ✓ Negative acute angles e.g. $(-\theta)$ are found in the 4th quadrant :
  - $\cos(-\theta) = \cos\theta$
  - $\sin(-\theta) = -\sin\theta$
  - $\tan(-\theta) = -\tan\theta$
- ✓ Negative obtuse angles, e.g. $[(-\theta - 180^0)$ or $(\theta - 360^0)]$ are found in the first, second, and third, quadrant.
- ✓ The cardinal angles $(90°, 180°\ and\ 360°)$ must be written first and be positive.

Methods to simplify negative angles are:

- **METHOD 1**: Add 360^0 to the angles of Sin and Cos until they become positive and add 180^0 to the angles of Tan to make them positive.
  - E.g. $\sin(-\theta - 180^0)$

    $= \sin(-\theta - 180^0 + 360^0)$

    $= \sin(-\theta + 180^0)$ [order does not matter e.g.$(-\theta + 180^0) = (180° - \theta)$

    $= \sin(180^0 - \theta)$

    $= \sin\theta$

- **METHOD 2 :** Rewrite the angle and make it a positive angle by taking out negative 1 as a common factor and thereafter reduce the angle using reduction formula:
  - E.g. $\sin(-\theta - 180^0) = \sin[-(\theta + 180^0)]$

    $= -\sin(180^0 + \theta)$

    $= -(-\sin\theta)$

    $= \sin\theta$

8. Trigonometric identities:

Square identity: $\sin^2\theta + \cos^2\theta = 1$

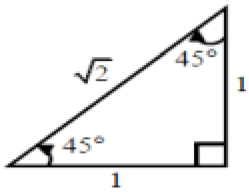
$\cos^2\theta = 1 - \sin^2\theta$

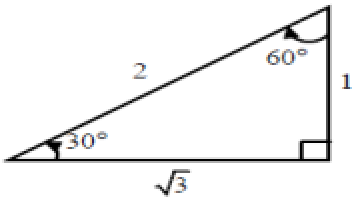
$\sin^2\theta = 1 - \cos^2\theta$

Quotient Identity: $\frac{sin\theta}{cos\,\theta} = tan\theta$

$\frac{cos\theta}{sin\theta} = \frac{1}{tan\theta}$

9. Special angles: *For most angles we need a calculator to calculate the values of sin, cos and tan. However, there are some angles we can easily work out the values for without a calculator as they produce simple ratios. The values of the trigonometric ratios for these special angles, as well as the triangles from which they are derived, are shown below* (Siyavula, 5.6)





t	30°	45°	60°
sin(t)	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos(t)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan(t)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

10. Compound identities:

- ✓ It deals with the addition and subtraction of different angles.
- ✓ If one of the angles is a cardinal angle $(90°, 180°\ and\ 360°)$, you may use reduction formula
- ✓ Sometimes when solving compound identities, the angle given may require sum or difference of special angles e.g. [$sin\ 15° = sin(60° - 45°)$ or $sin\ 15° = sin(45° - 30°)$]

cos (α – β) = cos α cos β + sin α sin β …… think "cos cos sin sin" ☺

Note the opposite signs

cos (α + β) = cos α cos β – sin α sin β

----

sin (α – β) = sin α cos β – cos α sin β …… think "sin cos cos sin" ☺

Note the same signs

sin (α + β) = sin α cos β + cos α sin β

The following compound identity proofs are examinable:

PROOF:

$\cos(\alpha + \beta) = \cos[\alpha - (-\beta)] = \cos\alpha.\cos(-\beta) + \sin\alpha.\sin(-\beta)$

$= \cos\alpha.\cos\beta + \sin\alpha.(-\sin\beta)$

$= \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$

PROOF:

$\sin(\alpha + \beta) = \cos[90° - (\alpha + \beta)] = \cos[90° - \alpha - \beta] = \cos[(90° - \alpha) - \beta]$

$= \cos(90° - \alpha).\ \cos(\beta) + \sin(90° - \alpha).\ \sin(\beta)$

$= \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$

PROOF:

$\sin(\alpha - \beta) = \cos[90° - (\alpha - \beta)] = \cos[90° - \alpha + \beta] = \cos[(90° + \beta) - \alpha]$

$= \cos(90° + \beta).\ \cos\alpha + \sin(90° + \beta).\ \sin\alpha$

$= -\sin\beta.\cos\alpha + \cos\beta.\sin\alpha$

$= \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

11. Double angles:

- ✓ It deals with addition of angles which are of the same value
- ✓ It deals with the relationship between $sine\ and\ cosine$
- ✓ When you substitute check the ratio next to $cos2x$, if it is $cosx$ substitute with $2cos^2x - 1$
- ✓ When you substitute check the ratio next to $cos2x$, if it is $sinx$ substitute with $1 - 2sin^2x$
- ✓ When you substitute check the ratio next to $cos2x$,if is $+1$ substitute with $cos^2x - 1$ or if is -1 substitute with $1 - 2sin^2x$
- ✓ When you substitute check the ratio next to $cos2x$, if it is $cosx\ and\ sinx$, substitute with $cos^2x - sin^2x$

NOTE!

$$\sin 2\alpha = \sin(\alpha + \alpha)$$

$$\sin 2\alpha = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$

$$\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$$

$= 2(\sin\alpha)(\cos\alpha)$ … (note that $\sin\alpha$ and $\cos\alpha$ are numbers, or factors in this case)

$$\cos 2\alpha = \cos(\alpha + \alpha)$$

$$\cos 2\alpha = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

$$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$$

$\cos 2\alpha = (1 - \sin^2\alpha) - \sin^2\alpha$ OR $\cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$

$\boxed{\cos 2\alpha = 1 - 2\sin^2\alpha}$ $\boxed{\cos 2\alpha = 2\cos^2\alpha - 1}$

$$2\cos^2\theta - 1 = \cos 2\theta$$

Half of 2θ ***double of θ***

$$2\cos^2\frac{\theta}{2} - 1 = \cos\theta$$

Half of θ ***double of*** $\boldsymbol{\frac{\theta}{2}}$

$$2\cos^2\frac{\theta}{2} - 1 = \frac{x}{r}$$

OR

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

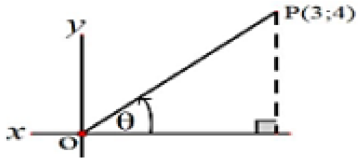
12. Tips on solving different trigonometric questions:

12.1. Questions requiring the use of a diagram:

- ✓ Firstly, make the trig ratios be subject of the formula and identify $r, x, and\ y$
- ✓ Check the quadrant.
- ✓ Draw the right-angled triangle and the line must lie on the x-axis (perpendicular to the x-axis), so that the x-axis is the base of the triangle.
- ✓ Use the Pythagoras to calculate the unknown side.
- ✓ Then simplify any question, take note of the double and compound angles.

Below are examples showing how to apply the tips:

1. In the diagram alongside P(3; 4) is a point in the Cartesian plane.
OP makes an angle θ with positive x- axis.
Without using a calculator, determine:



1.1 OP (1) **L1**

1.2 $\sin\theta + \cos\theta$ (2) **L1**

1.1 X = 3 $y = 4$ $r = ?$

$r^2 = (3)^2 + (4)^2$ **L1**

$r^2 = 25$

$r = \pm 5$

$\therefore r = 5$

$OP = 5\ units$

1.2 $\sin\theta + \cos\theta$ **L1**

$= \frac{y}{r} + \frac{x}{r}$

$$= \frac{4}{5} + \frac{3}{5}$$

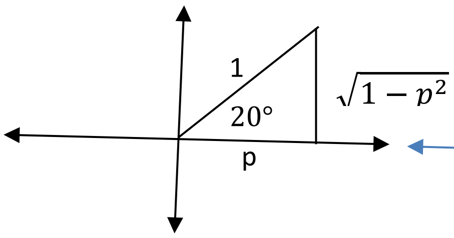
$$= \frac{7}{5}$$

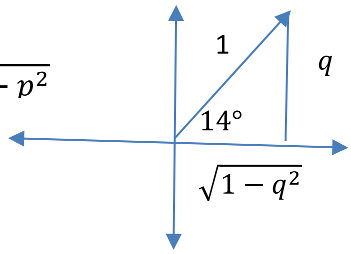
12. Given $\cos 20° = p$ and $\sin 14° = q$. Without using a calculator, calculate the value of the following in terms of p or q.

2.1 $\sin 20°$ (2) **L1** 2.2 $\cos 6°$ (4) **L3**

2.1 $sin20°$ (2) L1

$\sin 20 = \sqrt{1 - p^2}$





2.2 $cos6°$ (4) L3

$=\cos(20 - 14) = cos20cos14 + sin20sin14$

$= p.\sqrt{1 - q^2} + \sqrt{1 - p^2}.q$

12.2. Questions on expressing ratios as variable:

- ✓ Always draw the diagram on the first quadrant
- ✓ Use the Pythagoras to calculate the unknown side
- ✓ Then simplify any question, take note of double and compound angles
- ✓ Remember to convert the angle with special angles
- ✓ Find third angle by subtracting the given angle from 90°

12.3. Determining the invalid or undefined values of a function:

- ✓ Let the denominator equal to zero.
- ✓ If tangent is given, simply consider the asymptotes.
- ✓ Solve like general solutions.
- ✓ Substitute with integers, and then consider the domain.

12.4. Solving trigonometric equations and general solutions:

"*The periodicity of the trigonometric functions means that there are an infinite number of positive and negative angles that satisfy an equation. If we do not restrict* *the solution, then we need to determine the general solution to the equation. We know that the sine and cosine functions have a period of 360° and the tangent function has a period of 180°"* **(EMCGH)**

Tips:

- ✓ Simplify the equation using algebraic methods and trigonometric identities.
- ✓ Find factors to solve the general solutions.
- ✓ Determine the reference angle (use a positive value).
- ✓ If the angles are not equal and the co-efficient is 1, we use the co-function/angle to one of the sides. If the co-efficient is different then you must expand to give us tan
- ✓ For trinomial functions you must factorize by expanding or using quadratic function.
- ✓ For $\cos$ $and\ sin$ use $x = ref. < + K.360° \ K \in Z$
- ✓ For $tan\ x = ref. < + K.180° \ K \in Z$

13. Trigonometry functions:

Trigonometric graphs are about the relationships, simplification and determining points of intersection by solving equations (specific solutions). Although characteristics of the graphs should not be excluded.

Tips:

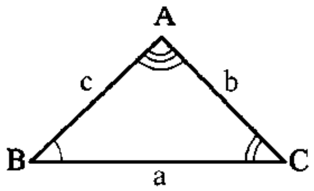
- ✓ Know how to sketch the graph of the original trigonometric functions: $[f(x) = \sin x, g(x) = \cos x\ , h(x) = \tan x]$
- ✓ Know how to perform transformation of graphs, that is, translation and reflection.
- ✓ Always know the domain, range, turning points, asymptotes and intercepts with axes of the trig graphs
- ✓ Know and understand the effects of parameters i.e. $\{a, k, p\ and\ q\}$ when given $f(x) = a \sin k(x + p) + q$.
- ✓ The effect of p on the shape of trigonometric graphs: If p is positive then the graph is shifted "p" degrees to the left, and if "p" is negative then the graph is shifted "p" degrees to the right
- ✓ Be able to find the equation from the graph, find the equation of any of the transformations of the graphs.

- ✓ The solution of trigonometric equations involving double and compound angles is important in this question. This is required when determining the coordinates of the points of intersections of two graphs, where the coordinates cannot be read off. If it is given that $\frac{AK}{AB} = \frac{3}{3}$, calculate the value of x.

14. TRIGONOMETRY: TWO AND THREE DIMENSIONS

When dealing with problems in right angled triangle trig ratios (SOHCAHTOA) can be used to solve for both angles and sides. But when dealing with non-right-angled triangles the trig rules are to be used, i.e Sine, Cosine and area rules.

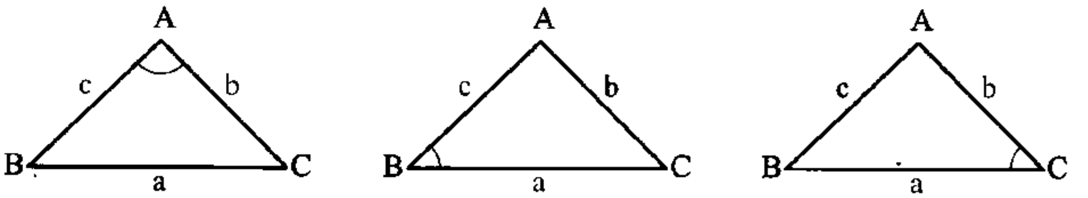
- **Conditions for using Sine Rule**:



$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}} \quad \textbf{or} \quad \frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$$

  - Use the Sine Rule when given two sides and an angle that is not between the given sides of if two angle and a side are given.
  - When using the Sine Rule place whatever you are looking for (angle or side) as the numerator of the fraction

- **Conditions for using Cosine Rule**

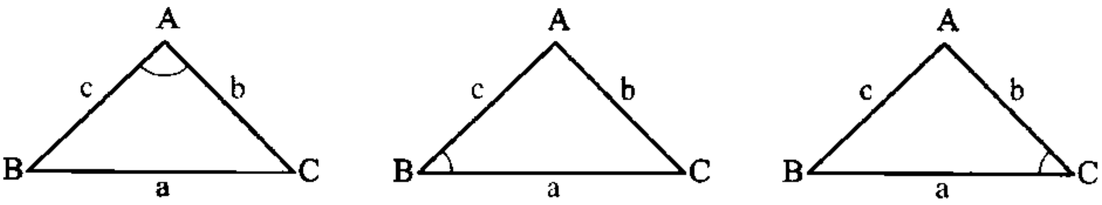


Angle A known Angle B known Angle C known

$$a^2 = b^2 + c^2 - 2(b)(c)\cos\hat{A} \quad b^2 = a^2 + c^2 - 2(a)(c)\cos\hat{B} \quad c^2 = a^2 + b^2 - 2(a)(b)\cos\hat{C}$$

  - Use the Cosine rule if you have a triangle where you are given **all three sides** or **two sides and included angle**.
  - If a side of a triangle has a square root in it, the Cosine Rule is the most likely rule to use.

- **Conditions for using Area rule:**



Angle A known: $A_{\Delta ABC} = \frac{1}{2} b\,c \sin \hat{A}$

Angle B known: $A_{\Delta ABC} = \frac{1}{2} a\,c \sin \hat{B}$

Angle C known: $A_{\Delta ABC} = \frac{1}{2} a\,b \sin \hat{C}$

  - Use it when given two sides and included angle.
  - When given area of a triangle and required to determine the side or an angle of a non-right-angled triangle.

15. Common errors committed by learners and examination guidelines:

15.1. Common errors committed by learners:

1. Many learners are unable to identify the correct quadrant in which a trig ratio belongs to, and this led to the use of incorrect values.

2. Learners cannot expand double angles and compound angles correctly, despite the identities given in the formula sheet.

3. Some learners have challenges of reducing/simplifying angles that are greater than 360^0.

4. When a question requires specific values, learners determine general solutions and stop there without further determining specific values.

5. Majority of learners cannot use the identity of Cos (A- B) to derive the identity for Sin(A-B).

6. With graphs, learners confuse the domain, range, and period.

7. Learners struggle with analyzing graphs.

8. Learners use wrong notations when interpreting graphs.

9. Learners cannot correctly apply area, sine and cosine rule, this is because they cannot analyze the given diagrams.

15.2 Examination guidelines:

15.2.1. The reciprocal ratios $cosec\theta, sec\theta\ and\ cot\theta\ can\ be\ used\ by\ candidates$ in the answering of problems but will not be explicitly tested.

15.2.2. The focus of trigonometric graphs is on the relationship, simplification and determining points of intersection by solving equations, although characteristics of the graphs should not be excluded.

16. Suggestions for improvement:

1. Educators must emphasize more on determining the correct quadrant in which the trig ratio belongs to by treating more questions requiring learners to draw their diagrams under given conditions.

2. Revision on grades 10 and 11 work is necessary.

3. Expose learners to questions on trigonometric ratios, involving combinations of compound angles, angles greater than 360^0 and co-ratios.

4. Learners should be given exercises to practice simplifying complex trigonometric expressions, proving identities, and solving complex trigonometric equations.

5. Teachers must make thorough revision on the mother functions on Sine, Cosine and Tan graphs, discussing their properties and how they are affected.

6. Teachers must devote the appropriate amount of time to 2D and 3D sections. This should allow learners to score the accessible marks in this section.

7. Teachers need to develop strategies to be used when solving right- angled triangles and non-right-angled triangles.

17. TRIGONOMETRY QUESTIONS ($\pm$50 MARKS)

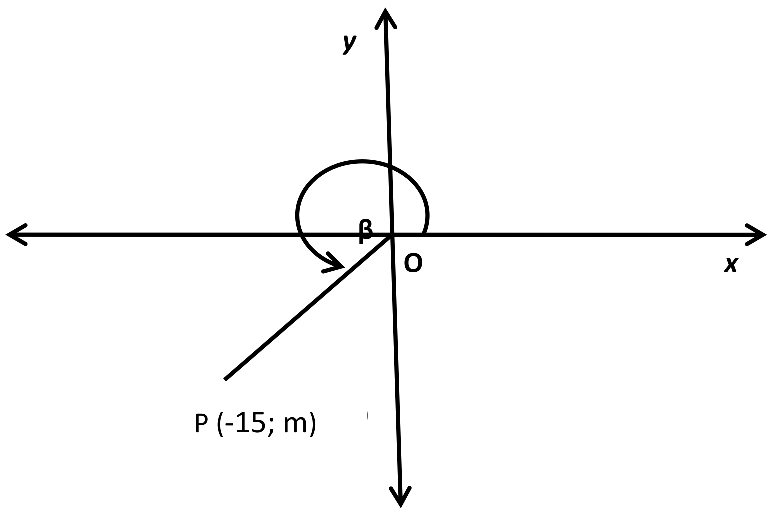
TRIG RATIO, IDENTITIES, REDUCTIONS, AND GENERAL SOLUTION.

Question 5

5.1 If $\cos\beta = \frac{p}{\sqrt{5}}$, where $p < 0$ and $\beta \in [180°; 360°]$, determine, using a diagram, an expression in terms of p for:

5.1.1 $\tan\beta$ (4) L2

5.2 In the diagram below, P (–15; *m*) is a point in the third quadrant and $17\cos\beta + 15 = 0$.



WITHOUT USING A CALCULATOR, determine the value of the following:

5.2.1 m (3) L2

5.2.2 $\sin\beta + \tan\beta$ (3) L2

5.3 Given: $sin\alpha = \frac{8}{17}$ where $90° \leq \alpha \leq 270°$

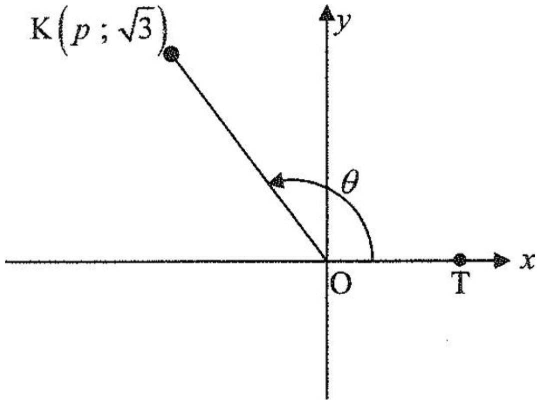
With the aid of a sketch and without the use of a calculator, calculate:

5.3.1 $tan\ \alpha$ (3) L2

5.3.2 $sin(90° + \alpha)$ (2) L2

5.3.3 $cos\ 2\alpha$ (3) L2

5.4 In the diagram, $K\ (p\ ;\sqrt{3})$ is a point in the 2nd quadrant. T is a point on the positive x-axis and obtuse $K\hat{O}T\ =\ \theta$.



5.4.1Write down the value of $\tan\theta$ in terms of p. (1) **L1**

5.4.2 If $\theta = 120°$, WITHOUT using a calculator, calculate the value of p. (3) **L2**

5.5 If cos 42^0 = t, WITHOUT using a calculator, write the following expressions in terms of t:

5.5.1 $sin48°$ (2)**L1**

5.5.2 $cos\ 84°$ (3) L2

5.6 If $\cos 34° = p$, WITHOUT using a calculator, write down the following in terms of p:

5.6 .1 $\cos 214°$ (2) **L2**

5.6 .2 $\cos 68°$ (2) **L2**

5.6 .3 $\tan 56°$ (4)

5.7 Simplify the following WITHOUT the use of a calculator:

5.7.1 $\dfrac{\tan(180°+x).\cos(360°-x)}{\sin(180-x).\cos(90°+x)+\cos(540°+x).\cos(-x)}$ (8) **L2**

5.7.2 $\dfrac{1-\cos 2x-\sin x}{\sin 2x-\cos x}$ (5) **L2**

5.7.3 $sin(90° - x).cos(180° - x) + \tan x.\cos(-x).\sin(180° + x)$ (7) **L2**

5.8 Do NOT use a calculator to answer this question. Show ALL calculations.

Prove that:

5.8.1 $\dfrac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 225°} = \dfrac{3}{2}$. (6) **L2**

5.8.2 $\cos 75° = \dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$ (4) **L2**

5.8.3 Simplify completely:

$\sin(90° - x)\cos(180° - x) + \tan x.\cos(-x)\sin(180° + x)$ (7) **L2**

5.9 Prove the following identities

5.9.2 $\cos 75° = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ (4) **L2**

5.10 Determine the general solution of

5.10.2 $\frac{\tan x - 1}{2} = -3$ (5)
L2

TRIGONOMETRIC FUNCTIONS

ACTIVITY 6A

1. Given: $f(x) = \cos 2x$ and $g(x) = \sin(x + 60°)$ for $x \in [-90°; 180°]$.

1.1. Sketch the graphs of f and g on the same system of axes for $x \in [-90°; 180°]$. Clearly show ALL intercepts with axes, points of intersections as well as turning points. (6) L2

1.2. Write down the period of $g\left(\frac{3}{2}x\right)$ (1) L1

1.3. Determine h if $h(x) = f(x - 45°) - 1$ (2) L2

ACTIVITY 6B

1. Consider $g(x) = -4\cos(x + 30°)$.

1.1. Write down the maximum value of $g(x)$ (1) L1

1.2. Determine the range of $g(x) + 1$. (2) L1

1.3. The graph of g is shifted 60^0 to the left and reflected about the x- axis to form a new graph h. Determine the equation of h in its simplest form. (3) L2

ACTIVITY 6C

1. Given: $f(x) = \sin(x - 30°)$ and $g(x) = \cos 3x$ for $x \in [-90°; 90°]$.

1.1. Write down the period of g. (1) L1

1.2. Draw the graphs of f and g on the same system of axes for $x \in [-90°; 90°]$. Clearly show ALL intercepts with axes, points of intersections as well as turning points and end points of both curves. (6) L2

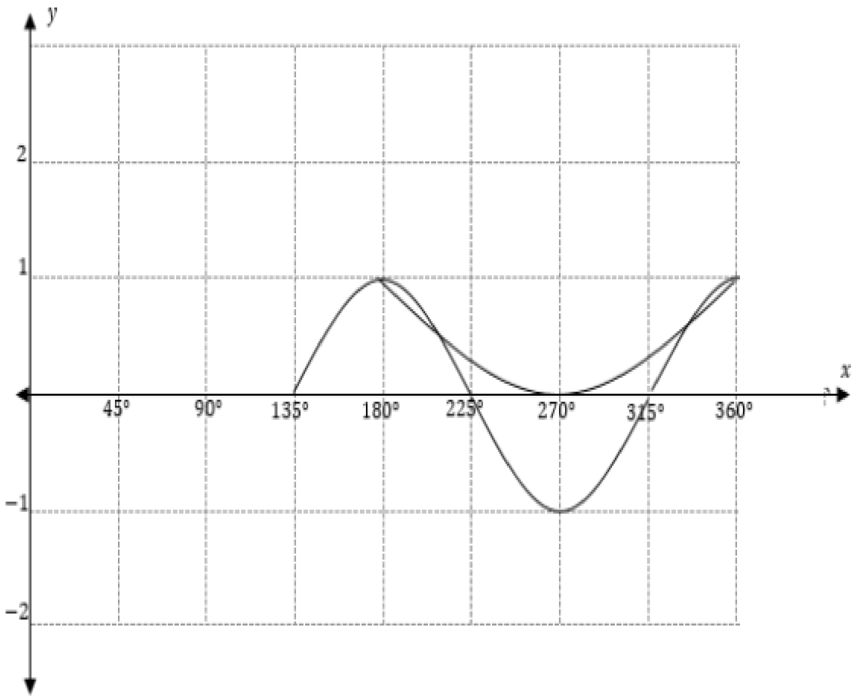
1.3. Use the graphs to determine the value(s) of x for $x \in [-90°; 90°]$, where :

1.3.1. $f(x) > g(x)$ (2) L2

1.3.2. $f(x).g(x) > 0$ (2) L2

1.4. Determine the range of $h(x) = 2f(x) - 1$ (2) L2

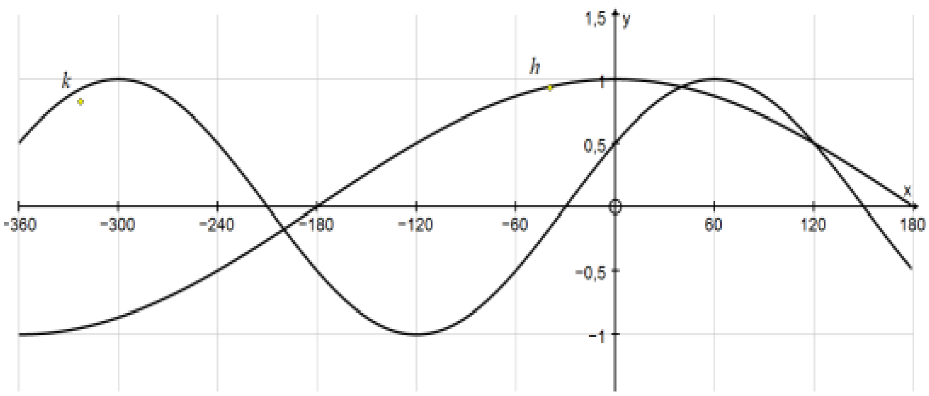
2. The graphs below represents $f(x) = 1 + \sin x$ and $g(x) = \cos 2x$ for $x \in [180°; 360°]$.



2.1. For which values of x will $f(x) \leq g(x)$ for $x \in [180°; 360°]$ (3) L2

ACTIVITY 6D

1. The diagram below shows the graphs of $h(x) = \cos px$ and $k(x) = \sin(x + q)$ for $x \in [-360°; 180°]$.

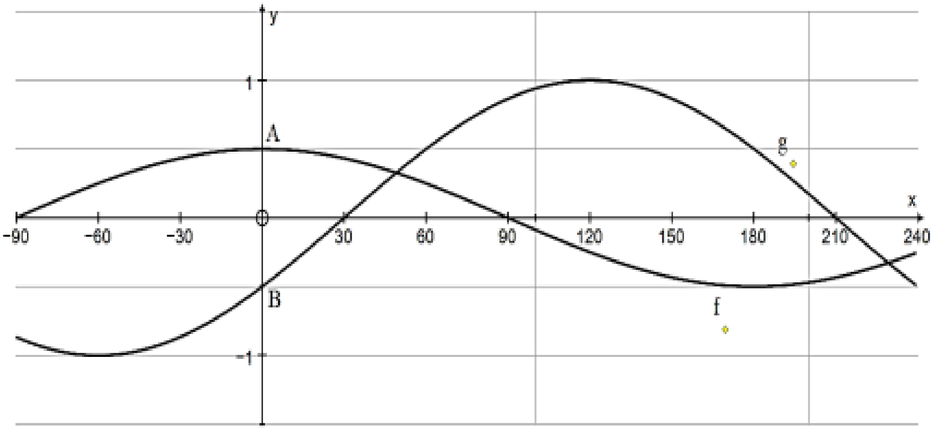


1.1. Write down the period of h. (1) L1

1.2. Determine the value of p and q. (2) L1

2. In the diagram below, the graphs of $f(x) = 2\sin x$ and

$g(x) = \cos(x - 30°)$ are drawn for $x \in [-180° ; 180°]$.

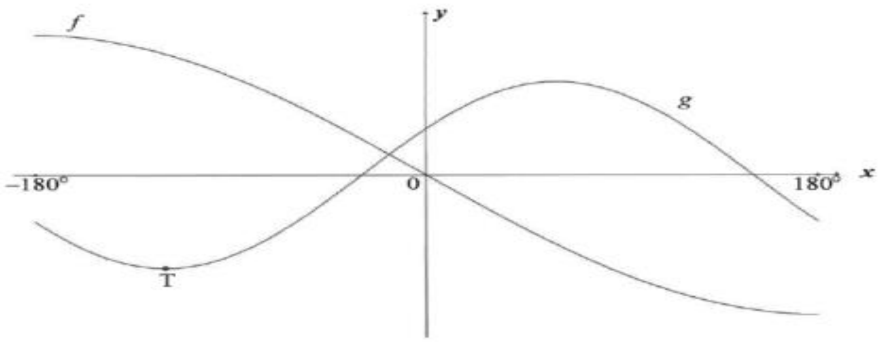


2.1. Write down the range of g. (1) L1

2.2. Write down the period of h if $h(x) = f\left(\frac{1}{2}x\right)$ (1) L2

ACTIVITY 6E

1. In the diagram, the graphs of $f(x) = -3\sin\frac{x}{2}$ and $g(x) = 2\cos(x - 60°)$ are drawn in the interval $x \in [-180°; 180°]$. $T(p; q)$ is a turning point of g with $p < 0$.



1.1. Write down the period of f. (1) L1

1.2. Write down the range of g. (2) L1

1.3. Use the graphs to determine the value(s) of x in the interval $x \in [-180°; 180°]$ for which:

1.3.1. $g(x) > 0$ (3) L2

ACTIVITY 6F

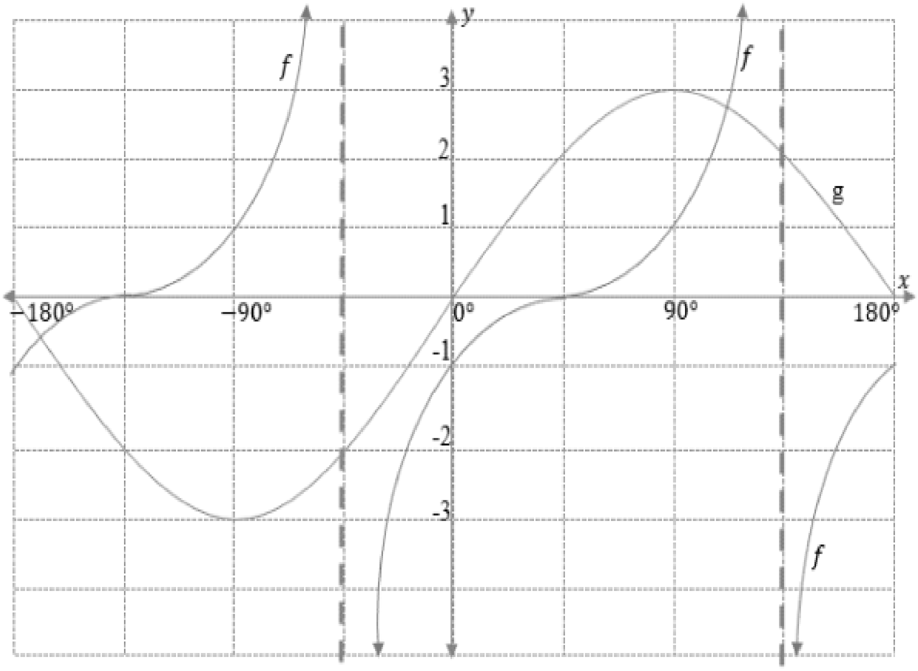
1. Given: $f(x) = 2\cos x$ and $g(x) = \tan 2x$

1.1. Sketch the graphs of f and g on the same system of axes for $x \in [-90°; 90°]$. Clearly show ALL intercepts with axes, points of intersections as well as turning points. (6) L2

1.2. Write down the period of $f\left(\frac{x}{2}\right)$ (2) L1

1.3. Write down the equations of the asymptotes of $g(x-25°)$, where $x \in [-90°; 90°]$. (2) L2

2. Sketched below are the graphs of the function $f(x) = \tan(x-45°)$ and $g(x) = 3\sin x$ for $x \in [-180°; 180°]$.



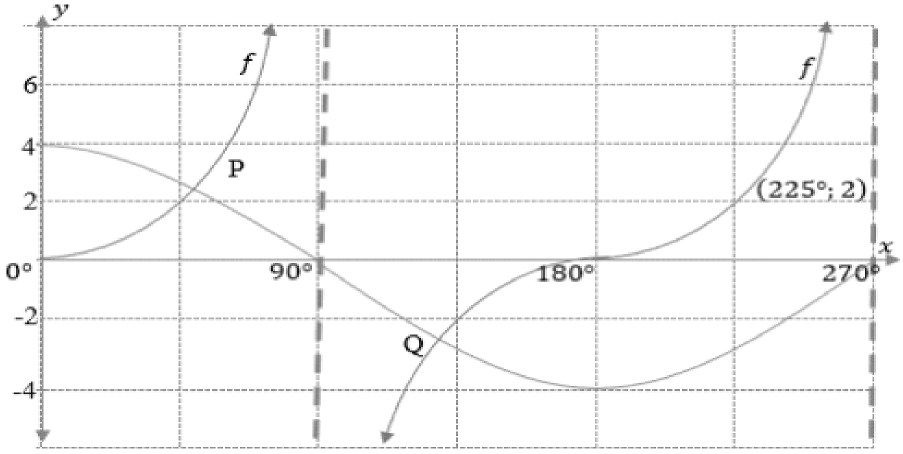
2.1. Write down the equations of the asymptotes of $y = f(x)$ for $x \in [-90°; 180°]$ (2) L1

2.2. Describe the transformation for graph of f to h if $h(x) = \tan(45° - x)$ (2) L2

ACTIVITY 6G

1. The graphs of the functions $f(x) = a\tan x$ and $g(x) = b\cos x$ for $0° \leq x \leq 270°$ are shown in the diagram below. The point $(225°; 2)$ lies on f.

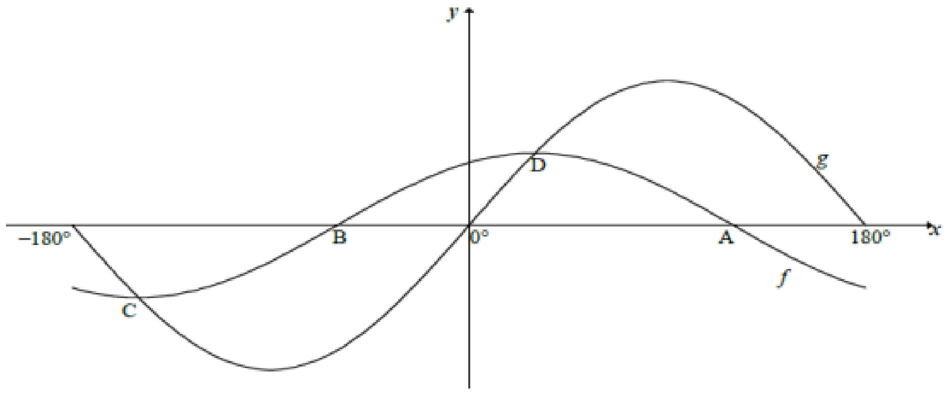
The graphs intersect at points P and Q.



1.1.	Determine the numerical values of a and b.	(4)	L2
1.2.	Determine the minimum value of $g(x) + 2$.	(2)	L2
1.3.	Determine the period of $f\left(\frac{1}{2}x\right)$	(2)	L2

ACTIVITY 6H

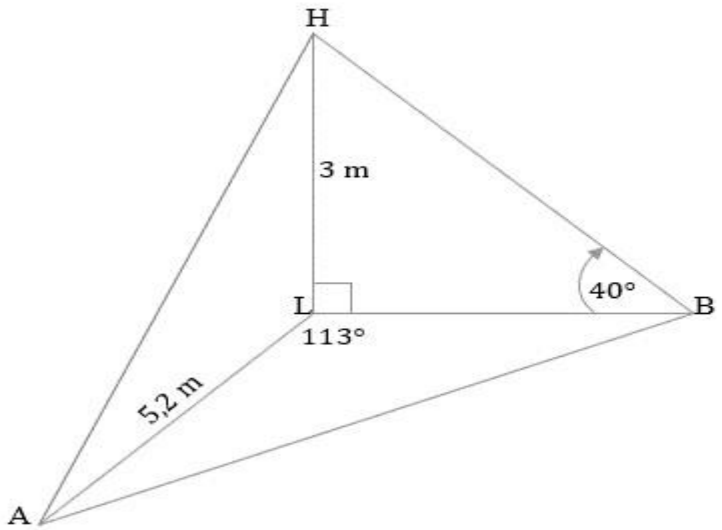
1. In the diagram, the graphs of $f(x) = \cos(x - 30°)$ and $g(x) = 2\sin x$ are drawn for the interval $x \in [-180°; 180°]$. A and B are the x-intercepts of f. The two graphs intersect at C and D, the minimum and maximum turning points respectively of f.



1.1.	Write down the coordinates of :		
	1.1.1. A	(1)	L1
	1.1.2. C	(2)	L2
1.2.	Determine the values of x in the interval $x \in [-180°; 180°]$, for which:		
	1.2.1. Both graphs are increasing	(2)	L2

TRIGONOMETRY 2D AND 3D
QUESTION 7A

A, B and L are points on the same horizontal plane. HL is a vertical pole of length 3 metres. AL = 5,2 m, the angle $A\hat{L}B = 113°$ and the angle of elevation of H from B is 40°.

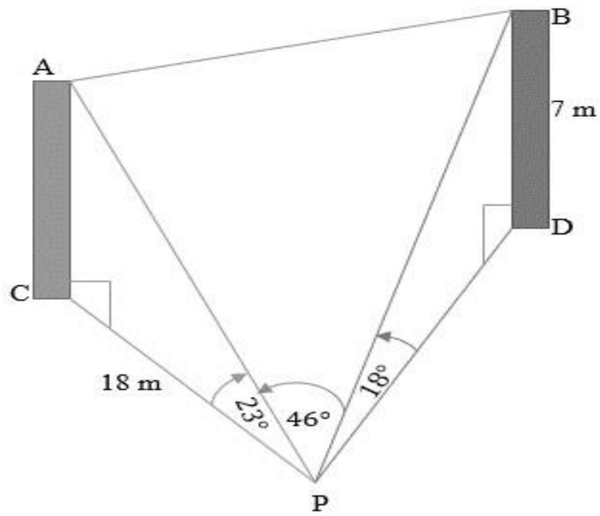


7.1 Calculate the length of LB. **(2)**

[L2]

QUESTION 7B

Thandi is standing at point P on the horizontal ground and observes two poles, AC and BD, of different heights, P, C and D are in the same horizontal plane. From P the angles of inclination to the top of the poles A and B are 23° and 18° respectively. Thandi is 18 m from the base of pole AC. The height of pole BD is 7 m.



Calculate, correct to TWO decimal places:

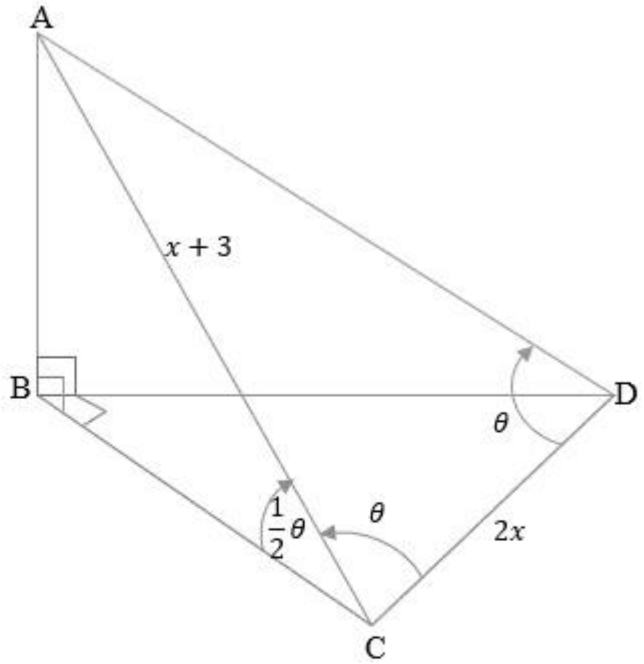
7.1 The distance from Thandi to the top of pole BD. (2) **[L2]**

7.2 The distance from Thandi to the top of pole AC. (2) **[L2]**

QUESTION 7C

A corner of a rectangular block of wood is cut off and shown in the diagram below. The inclined plane, that is, ΔACD, is and isosceles triangle having $A\hat{D}C = A\hat{C}D = \theta$.

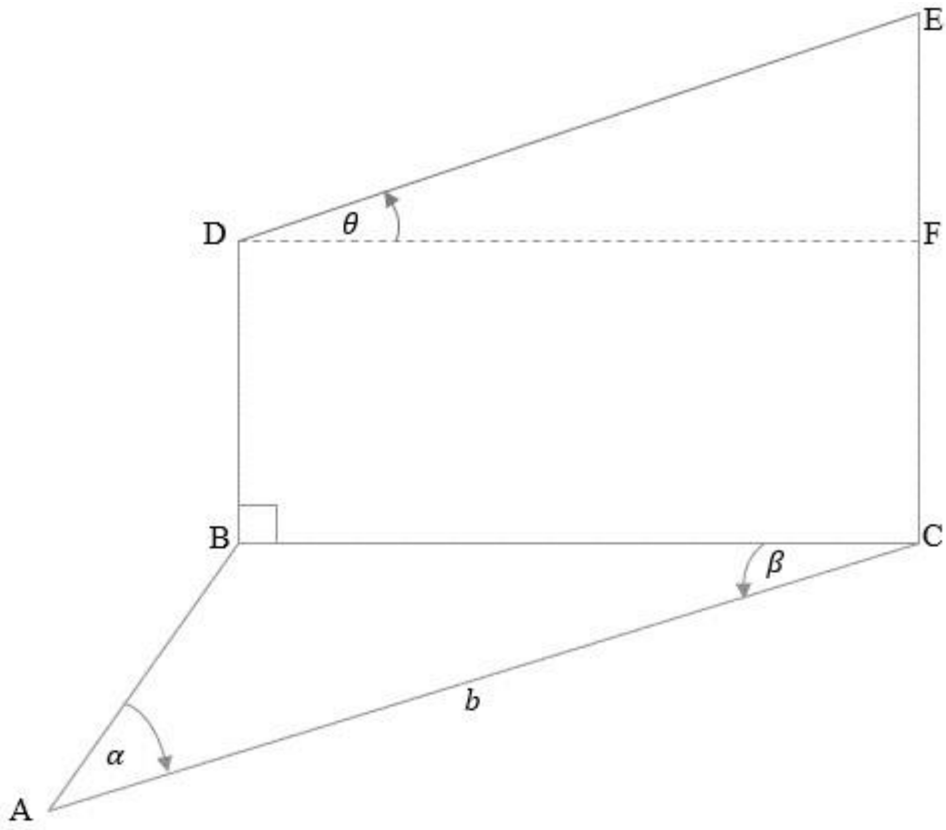
Also, $A\hat{C}B = \frac{1}{2}\theta$, AC = $x + 3$ and CD = $2x$.



7.1 Determine an expression for $C\hat{A}D$ in terms of θ. (1) **[L1]**

QUESTION 7D

In the diagram below A, B and C are three points in the same horizontal plane. D is vertically above B and E is vertically above C. The angle of elevation of E from D is θ. F is a point on EC such that DF ∥ BC. $B\hat{A}C = \alpha$, $A\hat{C}B = \beta$ and AC = b metres.

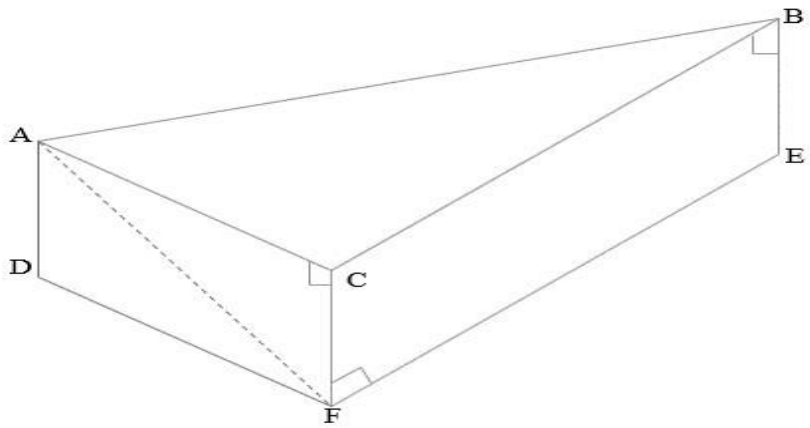


Given that $\mathrm{DE} = \dfrac{b \sin \alpha}{\sin(\alpha+\beta).\cos \theta}$.

Calculate DE if $b = 2\ 000$ metres, $\alpha = 43°$, $\beta = 36°$ and $\theta = 27°$. (3)

QEUSTION 7E

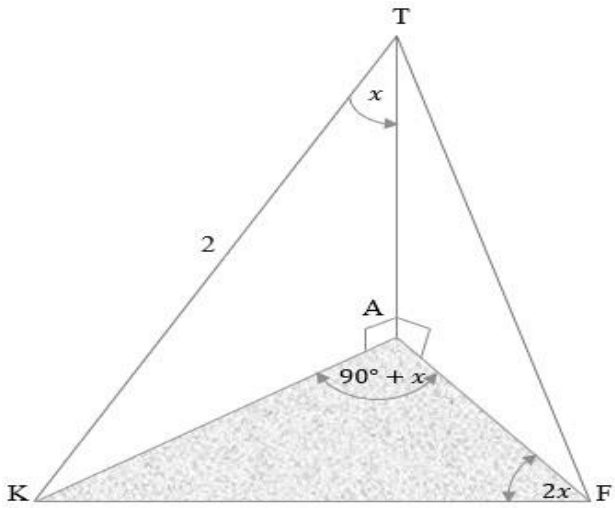
The figure below represents a triangular right prism with BA = BC = 5 units, $A\hat{B}C = 50°$ and $F\hat{A}C = 25°$.



7.1 Determine the area of ΔABC. (2)[**L2]**

QUESTION 7F

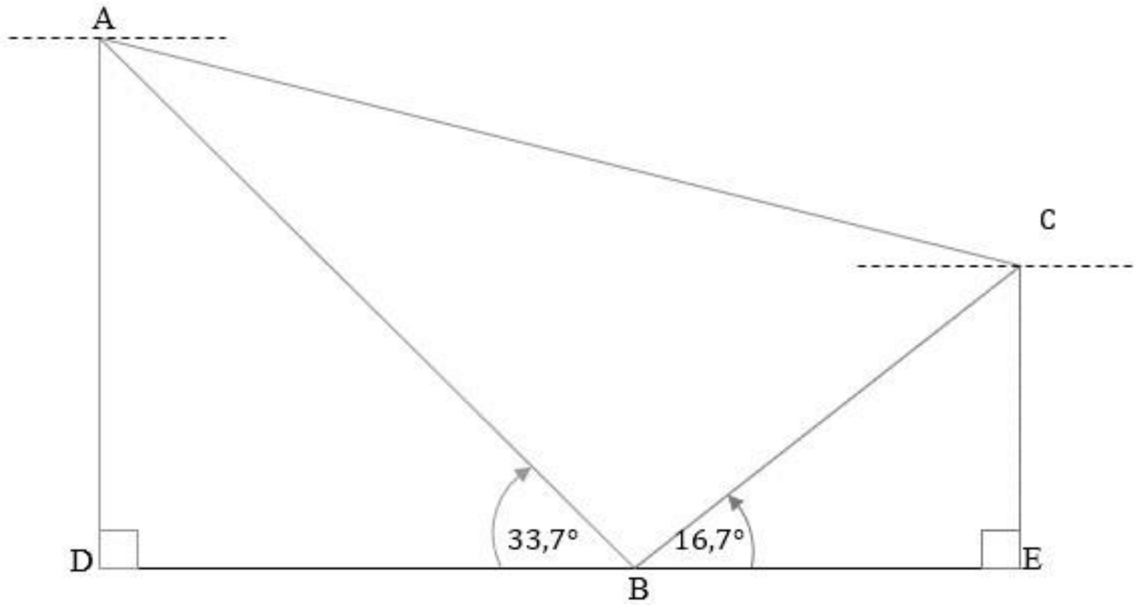
In the figure, points K, A and F lie in the same horizontal plane and TA represents a vertical tower. $A\hat{T}K = x$, $K\hat{A}F = 90° + x$ and $K\hat{F}A = 2x$ where $0° < x < 30°$. TK = 2 units.



7.1 Express AK in terms of $\sin x$. (2)[**L2**]

QUESTION 7G

In the diagram below, C is a point on one side of the Buffalo River and is 3 m above the water. A is a point on the other side of the river directly opposite C on the higher bank. B is a boat on the river. A, B and C are in the same vertical plane. The angle of depression of B from A is 33,7°. The angle of depression of C from A is 15,60° and B from C is 16,7°.



7.1 Calculate the length of BC. (3)[**L2**]

7.2 Calculate the length of AB. (3)**[L2]**

7.3 Calculate the length of AD. (3)[**L2**]

CAPRICORN NORTH

EUCLIDEAN GEOMETRY

WORKING MANUAL

TABLE OF CONTENT

EUCLIDEAN GEOMETRY AND MEASUREMENT

1. Measurement can be tested in the context of optimisation in calculus and two- and three-dimensional trigonometry.

2. Composite shapes could be formed by combining a maximum of TWO of the stated shapes.

3. The following proofs of theorems are examinable:
   - The line drawn from the centre of a circle perpendicular to a chord bisects the chord;
   - The line drawn from the centre of a circle that bisects a chord is perpendicular to the chord;
   - The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);
   - The opposite angles of a cyclic quadrilateral are supplementary;
   - The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment;
   - A line drawn parallel to one side of a triangle divides the other two sides proportionally;
   - Equiangular triangles are similar.

4. Corollaries derived from the theorems and axioms are necessary in solving riders:
   - Angles in a semi-circle
   - Equal chords subtend equal angles at the circumference
   - Equal chords subtend equal angles at the centre
   - In equal circles, equal chords subtend equal angles at the circumference
   - In equal circles, equal chords subtend equal angles at the centre.
   - The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral.
   - If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.
   - Tangents drawn from a common point outside the circle are equal in length.

5. The theory of quadrilaterals will be integrated into questions in the examination.

6. Concurrency theory is excluded.

4. ACCEPTABLE REASONS: EUCLIDEAN GEOMETRY

In order to have some kind of uniformity, the use of the following shortened versions of the theorem statements is encouraged.

4.1 ACCEPTABLE REASONS: EUCLIDEAN GEOMETRY (ENGLISH)

THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	∠'s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj ∠'s supp
The adjacent angles in a revolution add up to 360□.	∠'s round a pt **OR** □∠'s in a rev
Vertically opposite angles are equal.	vert opp ∠'s =
If AB ‖ CD, then the alternate angles are equal.	alt ∠'s ; AB ‖ CD
If AB ‖ CD, then the corresponding angles are equal.	corresp ∠'s ; AB ‖ CD
If AB ‖ CD, then the co-interior angles are supplementary.	co-int ∠'s ; AB ‖ CD
If the alternate angles between two lines are equal, then the lines are parallel.	alt ∠'s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp ∠'s =
If the co-interior angles between two lines are supplementary, then the lines are parallel.	coint ∠'s supp
TRIANGLES	
The interior angles of a triangle are supplementary.	∠ sum in Δ **OR** sum of ∠'s in Δ **OR** Int ∠'s Δ
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext ∠'s of Δ
The angles opposite the equal sides in an isosceles triangle are equal.	∠'s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal ∠'s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras **OR** Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras **OR** Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS **OR** S∠S

If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS **OR** ∠∠S
If in two right-angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS **OR** 90°HS

THEOREM STATEMENT	**ACCEPTABLE REASON(S)**
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt $\parallel$ to 2^{nd} side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line $\parallel$ one side of Δ□ **OR** prop theorem; name $\parallel$ lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	$\vert\vert\vert$ Δ's **OR** equiangular Δs
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height **OR** equal bases; equal height
CIRCLES	
The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	tan ⊥radius tan ⊥ diameter
If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.	line⊥ radius **OR** converse tan ⊥ radius **OR** converse tan ⊥diameter
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	line from centre to midpt of chord
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	line from centre to chord
The perpendicular bisector of a chord passes through the centre of the circle;	perp bisector of chord
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	∠ at centre = 2 ×∠ at circumference

The angle subtended by the diameter at the circumference of the circle is 90□.	∠'s in semi-circle **OR** diameter subtends right angle **OR** ∠'s in $\frac{1}{2}\odot$
If the angle subtended by a chord at the circumference of the circle is 90□, then the chord is a diameter.	chord subtends 90□ **OR** converse ∠'s in semi-circle
Angles subtended by a chord of the circle, on the same side of the chord, are equal	∠'s in the same seg
If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	line subtends equal ∠'s **OR** converse ∠'s in the same seg
Equal chords subtend equal angles at the circumference of the circle.	equal chords; equal ∠'s
Equal chords subtend equal angles at the centre of the circle.	equal chords; equal ∠'s
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal ∠'s

THEOREM STATEMENT	**ACCEPTABLE REASON(S)**
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal ∠'s
The opposite angles of a cyclic quadrilateral are supplementary	Opp ∠'s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	Opp ∠'s quad supp **OR** converse opp ∠'s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext ∠ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	ext ∠ = int opp ∠□ **OR** converse ext ∠ of cyclic quad
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt **OR** Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem **OR** between line and chord
QUADRILATERALS	
The interior angles of a quadrilateral add up to 360□.	sum of ∠'s in quad
The opposite sides of a parallelogram are parallel.	opp sides of ‖m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are ‖
The opposite sides of a parallelogram are equal in length.	opp sides of ‖m

If the opposite sides of a quadrilateral are equal , then the quadrilateral is a parallelogram.	opp sides of quad are = **OR** converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp ∠'s of ||m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp ∠'s of quad are = **OR** converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of ||m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other **OR** converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and ||
The diagonals of a parallelogram bisect its area.	diag bisect area of ||m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect
The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

Common misconceptions

- Learners give incorrect or incomplete reasons.
- Learners name angles incorrectly.
- Learners make many irrelevant statements.
- Learners fail to mention the parallel lines in the reason.
- Learners unable to identify t he corresponding sides and angles.
- Learners are not clear with the difference between the value of a ratio and a length.
- Learners fail to recall properties of quadrilaterals required.
- Learners are not able to use correct reasons for converses.
- Learners fail to understand given information and linking it to the diagram (Rushing to
- answer questions)
- Making assumptions
- Learners fail to link questions.
- Little understanding of the difference between a theorem and its converse
- Little understanding between congruency and similarity.
- Lack of content knowledge of key axioms and corollaries

Suggestions for improvement

- **Understand where learners can get 'easy' marks in the topic.**

We show how marks are allocated and where learners can get "easy" marks in each topic. Every mark count, so it's silly not to know where to find the "easy" marks.

- **Know what will be examined.**

We break down each topic and highlight what learners will be examined on. This guide also reminds teachers and learners of the Grade 11 topics learners will be examined on, which are often forgotten by the time they write Grade 12 exams.

- **Focus on what learners need to understand.**

This guide shows teachers what learners need to understand in the topic and gives pointers on how to master it. It further identifies the formulas relevant to each section, as well as the key words to look out for when answering questions – these will assist learners in knowing what is examined and how they should approach answering the questions.

- **Learn from the mistake's others made.**

The Department of Basic Education publishes a diagnostic report after releasing the final Grade 12 results. This gives us valuable information on common mistakes learners made in answering exam papers – if we take note of these, we can avoid our learners making the same mistakes.

- **Know and understand the calculator.**

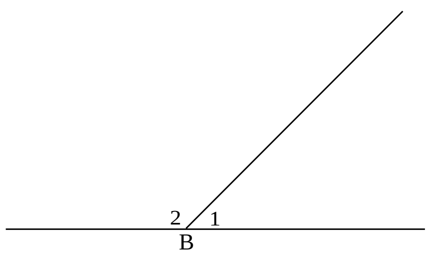
Learners often lose marks simply because they don't know how to use their calculators, to arrive at answers. Here too, teachers can assist learners or use the support platforms offered with this guide, to explain how to use calculators correctly.

Gr 8 – 12 GEOMETRY (THEORY)

LINES AND ANGLES (Grade 8)

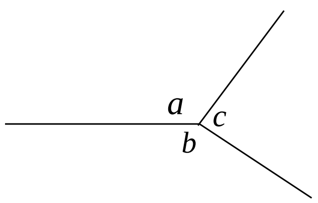
Adjacent supplementary angles

In the diagram, $\hat{B}_1 + \hat{B}_2 = 180°$



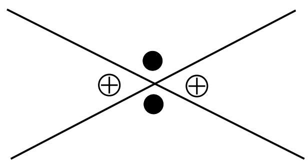
Angles round a point

In the diagram, $a + b + c = 360°$



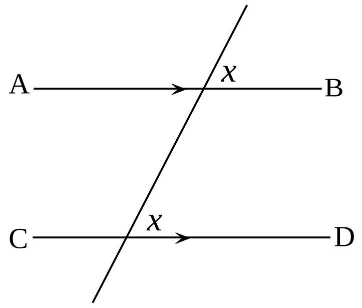
Vertically opposite angles

Vertically opposite angles are equal.



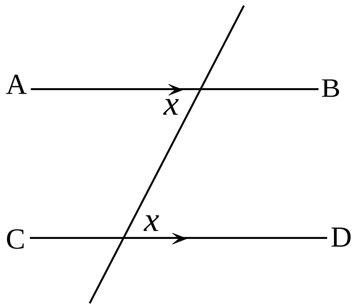
Corresponding angles

If AB||CD, then the corresponding angles are equal.



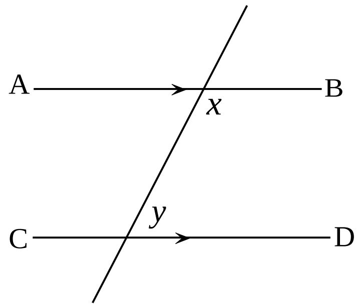
Alternate angles

If AB||CD, then the alternate angle are equal.



Co-interior angles

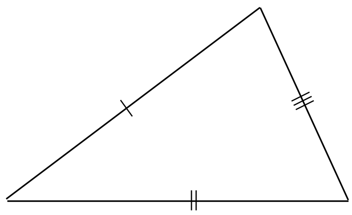
If AB||CD, then the co-interior angles add up to $180°$, i.e. $x + y = 180°$



TRIANGLES (Grade 8)

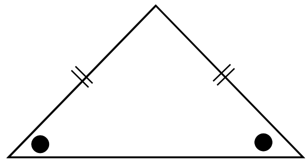
There are four kinds of triangles:

Scalene Triangle
No sides are equal in length



Isosceles Triangle
Two sides are equal

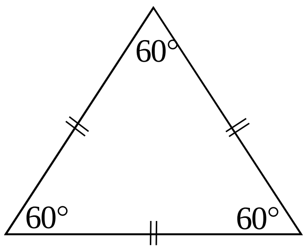
Base angles are equal



Equilateral Triangle \

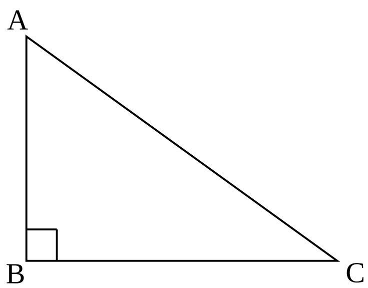
All three sides are equal

All three interior angles are equal

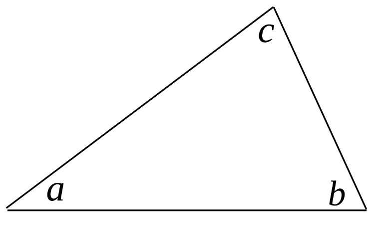


Right-angled triangle

One interior angle is $90°$

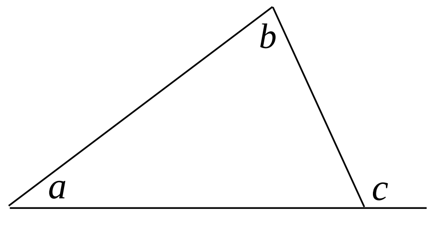


Sum of the angles of a triangle



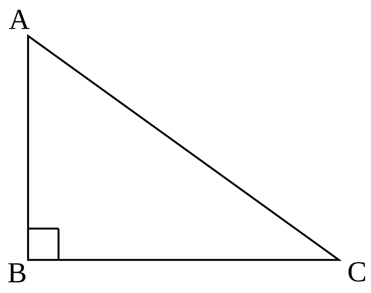
$a+b+c=180°$

Exterior angle of a triangle



$c=a+b$

The Theorem of Pythagoras



$AC^2 = AB^2 + BC^2$

or

$AB^2 = AC^2 - BC^2$

or

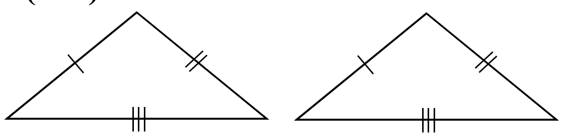
$BC^2 = AC^2 - AB^2$

<u>Congruency of triangles (four conditions) (Grade 9)</u>

Symbol for congruence (≡)

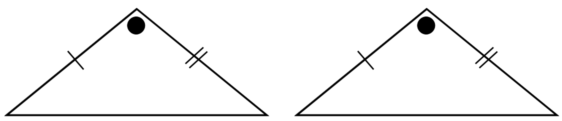
Condition 1 (SSS)

Two triangles are congruent if three sides of one triangle are equal in length to the three sides of the other triangle.



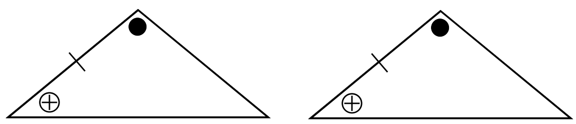
Condition 2 (S∠S)

Two triangles are congruent if two sides and the included angle are equal to two sides and the included angle of the other triangle.



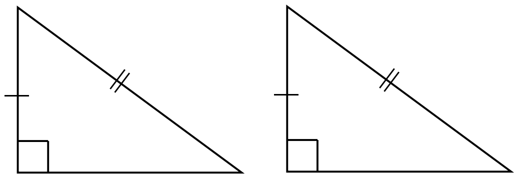
Condition 3 (∠S∠ or ∠∠S or S∠∠)

Two triangles are congruent if two angles and one side are equal to two angles and one corresponding side of the other triangle.



Condition 4

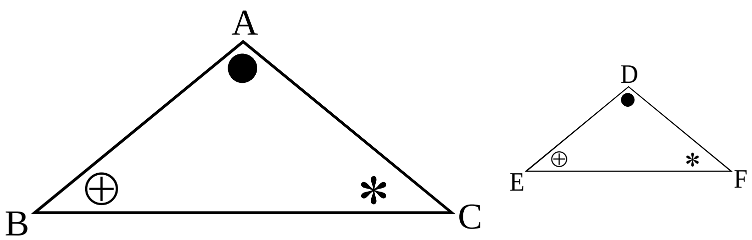
Two right-angled triangles are congruent if the hypotenuse and a side of the one triangle is equal to the hypotenuse and a side of the other triangle.



Similar Triangles (Grade 9)

If two triangles are similar then they are equiangular and their corresponding sides are in the same proportion.

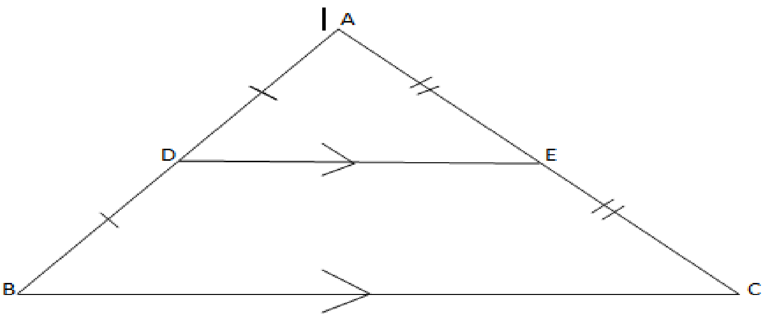
If $\Delta ABC ||| \Delta DEF$, then $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$



The mid-point theorem (no proof needed) (Grade 10)

If in ΔABC , D is the midpoint of AB and E the midpoint of AC, then

DE $\|$ BC and DE is ½ BC.

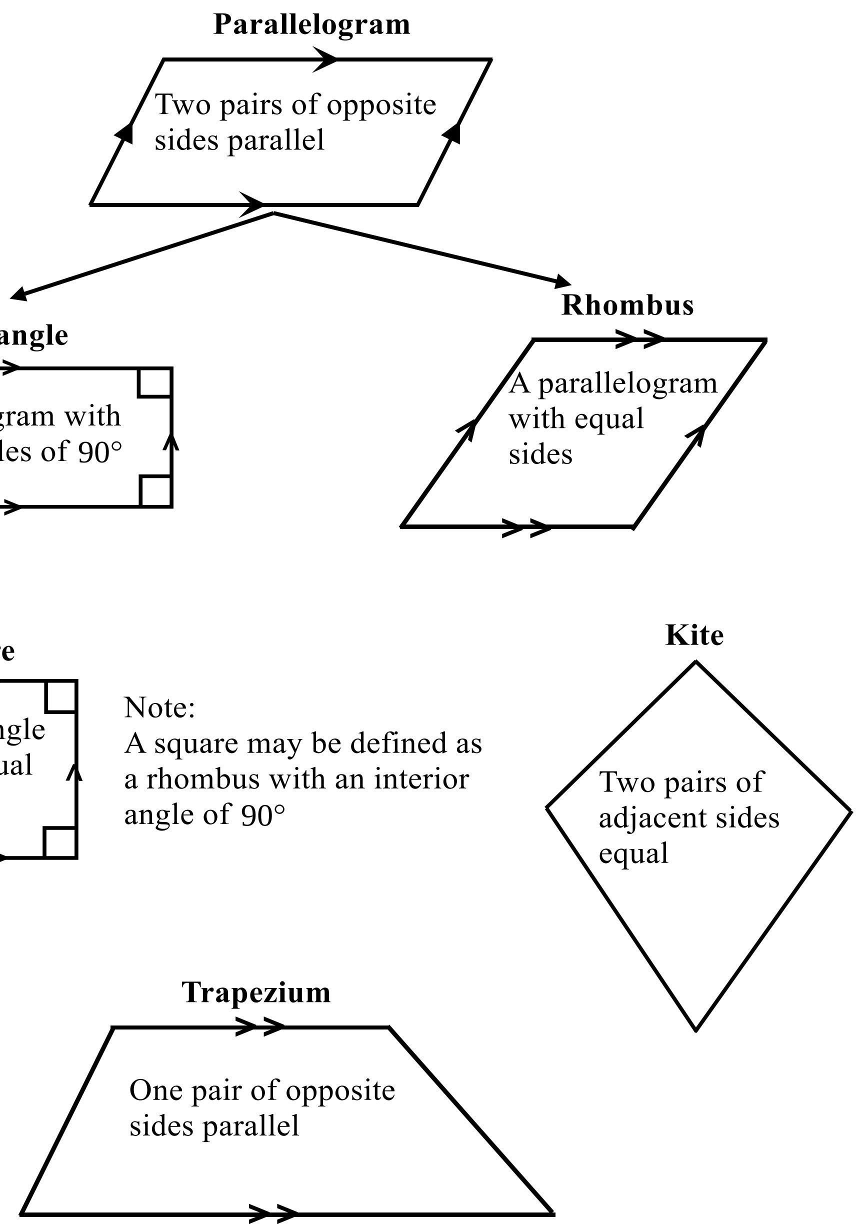


QUADRILATERALS (Grade 10)

A **polygon** is a closed two-dimensional figure with three or more straight sides.

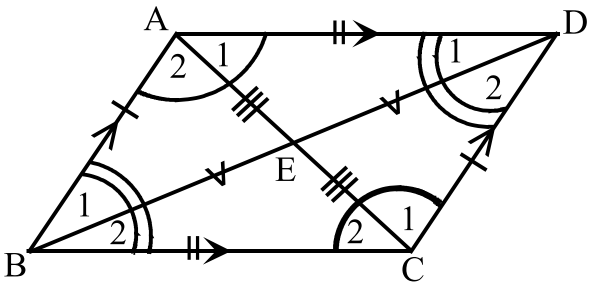
A **quadrilateral** is a polygon with four straight sides.

Definitions of quadrilaterals



Parallelograms (Grade 10)

If ABCD is a parallelogram, you may assume the following properties:



AD||BC ; AB||DC
AD = BC ; AB = DC
AE = EC ; BE = ED
$\hat{D}_1 = \hat{B}_2$; $\hat{D}_2 = \hat{B}_1$; $\hat{C}_1 = \hat{A}_2$; $\hat{C}_2 = \hat{A}_1$
$\hat{A} = \hat{C}$; $\hat{B} = \hat{D}$

In order to prove that a quadrilateral is a parallelogram, you will need to prove at least one of the following:

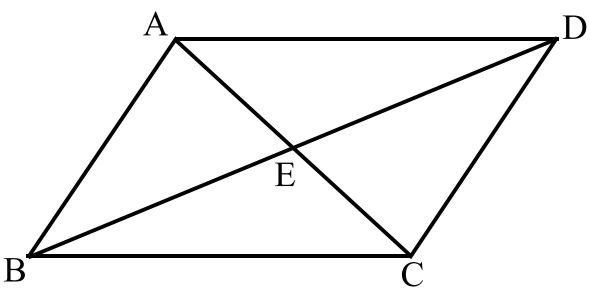
a) Prove both pairs of opposite sides parallel **or**

b) Prove both pairs of opposite sides equal **or**

c) Prove the diagonals bisect each other **or**

d) Prove both pairs of opposite angles equal **or**

e) Prove one pair of opposite equal **and** parallel

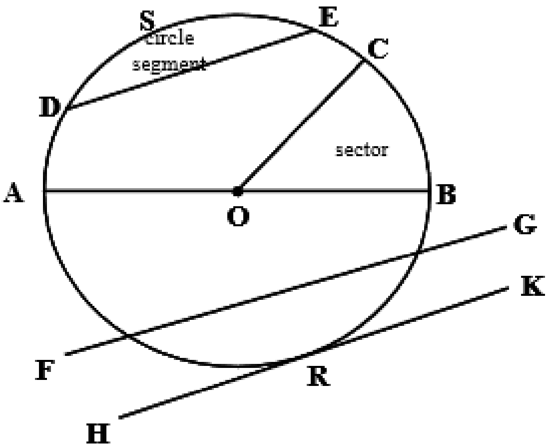


AD||BC and AB||DC	Opp sides ||
AD = BC and AB = DC	Opp sides =
AE = EC and BE = ED	Diagonals bisect
$\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$	Opp angles =
AB||DC and AB = DC	
AD||BC and AD = BC	

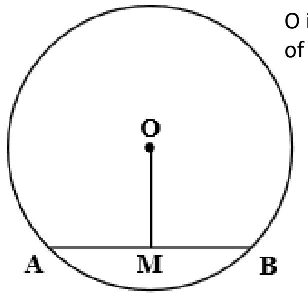
CIRCLE GEOMETRY (Grade 11)

TERMINOLOGY

O = centre
OC = radius
AB = diameter
DE = chord
DSE = arc
FG = secant
HK = tangent
R = point of tangency

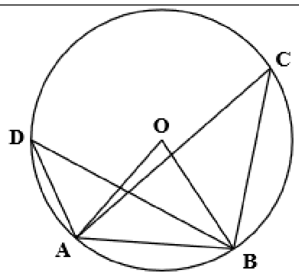


SUMMARY OF THEOREMS



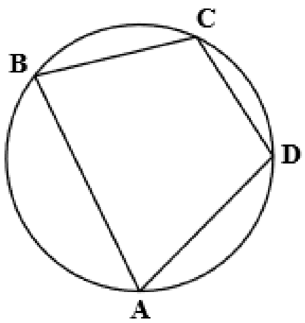
O is the centre of the circle

$\mathbf{OM \perp AB \;\Leftrightarrow\; AM = MB}$



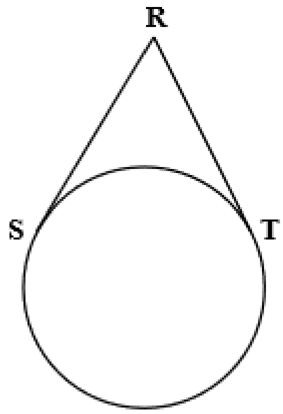
O is the centre of the circle

$2A\hat{D}B = A\hat{O}B = 2A\hat{C}B$

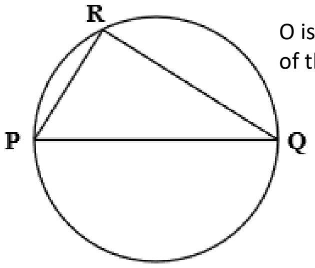


$A\hat{B}C + A\hat{D}C = 180°$

$B\hat{A}D + B\hat{C}D = 180°$

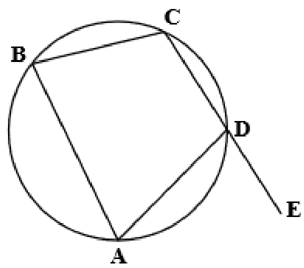


RS and RT tangents $\Rightarrow$ RS = RT

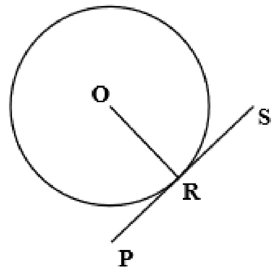


O is the centre of the circle

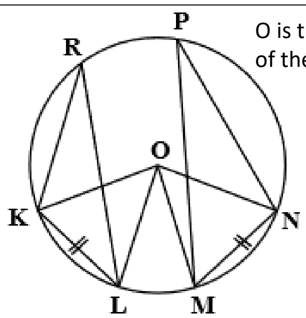
PQ a diameter $\Leftrightarrow P\hat{R}Q = 90^\circ$



$A\hat{D}E = A\hat{B}C$

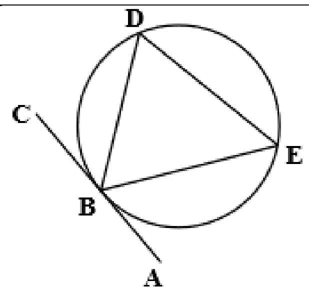


PR a tangent $\Leftrightarrow OR \perp PS$

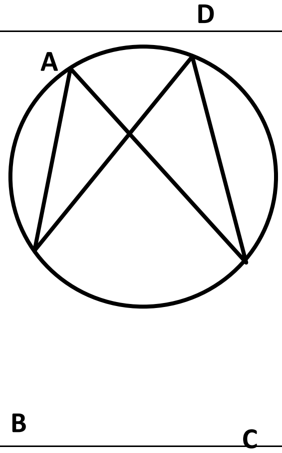


O is the centre of the circle

$K\hat{R}L = M\hat{P}N \Leftrightarrow KL = MN \Leftrightarrow K\hat{O}L = M\hat{O}N$



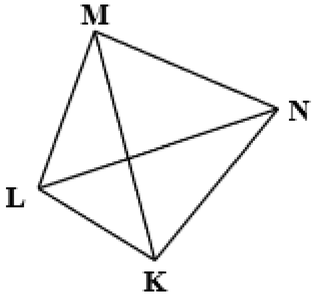
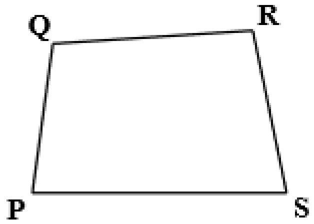
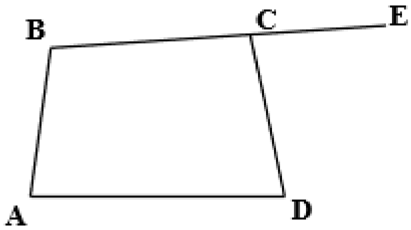
CA a tangent $\Leftrightarrow C\hat{B}D = B\hat{E}D$



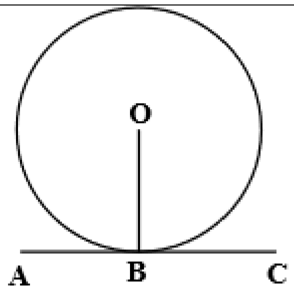
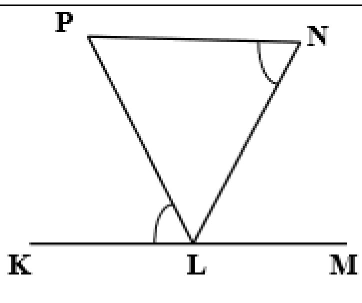
$\hat{A} = \hat{D}$
and
$\hat{B} = \hat{C}$

Formal proofs of these four theorems must be learned

HOW TO PROVE THAT A QUADRILATERAL IS A <u>CYCLIC</u> QUADRILATERAL

	**KLMN** a cyclic quadrilateral $\Leftrightarrow \mathbf{K\hat{L}N = K\hat{M}N}$
	**PQRS** a cyclic quadrilateral $\Leftrightarrow \mathbf{\hat{Q} + \hat{S} = 180^\circ}$
	**ABCD** a cyclic quadrilateral $\Leftrightarrow \mathbf{E\hat{C}D = B\hat{A}D}$

HOW TO PROVE THAT A LINE IS A <u>TANGENT</u> TO A CIRCLE

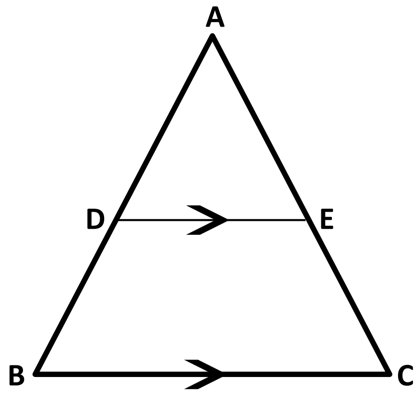
	**AC** a tangent $\Leftarrow \mathbf{O\hat{B}A = 90^\circ = O\hat{B}C}$
	**KM** a tangent to the **circle** through **PLN** $\Leftarrow \mathbf{P\hat{L}K = P\hat{N}L}$

RATIO & PROPORTION (Grade 12)

Theorem 1: (formal proof required)

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

line || one side of Δ



$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\frac{AB}{AD} = \frac{AC}{AE}$$

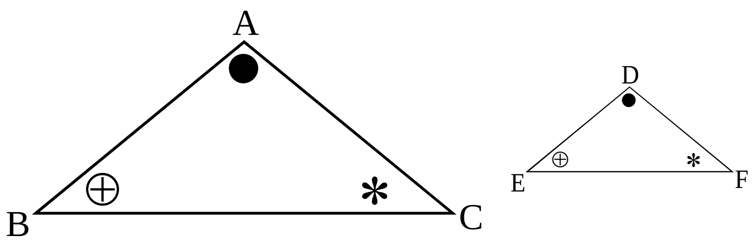
$$\frac{AB}{BD} = \frac{AC}{CE}$$

SIMILARITY (grade 12)

Theorem 2: (formal proof required)

If the corresponding angles of two triangles are equal, then the corresponding sides are in proportion (and consequently the triangles are similar).

||| Δs **OR** equiangular Δs



If: $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$

Then: $\Delta ABC ||| \Delta DEF$

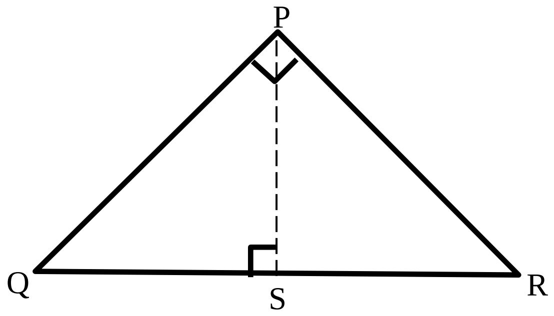
Therefore: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

THE THEOREM OF PYTHAGORAS (Grade 12)

Theorem 3: (no formal proof required)

In a right-angled triangle, the square of the hypotenuse side is equal to the sum of the squares of the other two sides.

Pythagoras **OR** Pyth.



$\text{QR}^2 = \text{PQ}^2 + \text{PR}^2$

- ❖ By using the same diagram, the following ***important*** deductions are made:

1. ΔPQR lll ΔSQP lll ΔSPR and

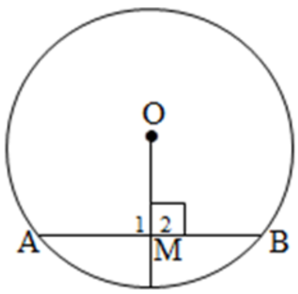
2. ΔPQS lll ΔRQP: $QP^2 = QS\,.\,QR$
ΔPRS lll ΔQRP: $RP^2 = RS\,.\,RQ$
ΔPQS lll ΔRPS: $PS^2 = QS\,.\,RS$

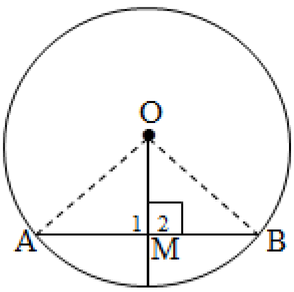
Perp. from the right $<^{ed}$ vertex to the hyp.

EXAMINABLE EUCLIDEAN GEOMETRY PROOFS IN GRADE 12

GRADE 11

1.	The line segment from the centre of a circle perpendicular to a chord will bisect the chord.





Constr: Draw OA and OB

In ΔOAM and ΔOBM

$\hat{M}_1 = \hat{M}_2 = 90°$ [**gi**ve]

OA = OB [**ra**dii]

OM = OM [**co**mmon]

∴ ΔOAM ≡ ΔOBM [RHS]

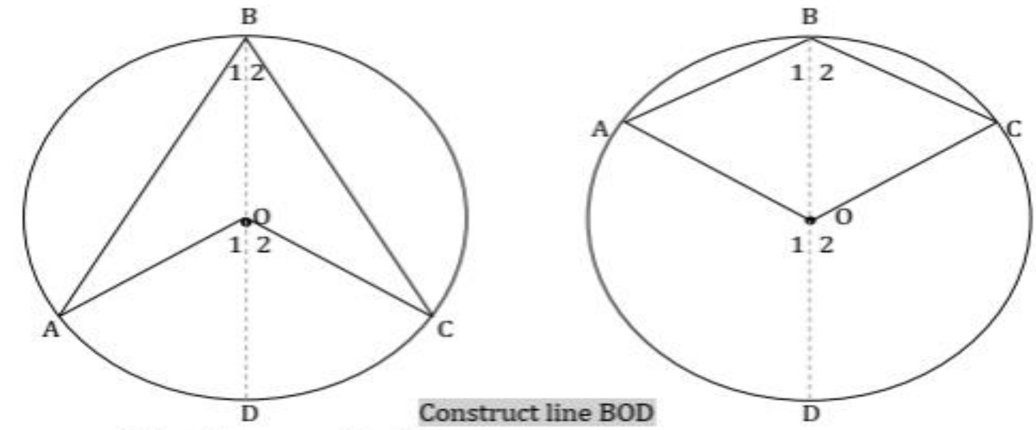
∴AB = MB

HINT : **GIRACO (G-GIVEN, R-RADII,C-COMMON)**

2. In the diagram, O is the centre of the circle with A, B and C drawn on the circle. Prove the theorem that states that an angle at the centre of a circle is twice the angle on the circumference of the circle, subtended by the same chord/arc.

Angle at the centre is equal to 2 times the angle at the circumference

Prove that the angle at the centre is equal to 2 times the angle at the circumference.



Construct line BOD

$OA = OB$ (radii)

$A = B_1$ ($\angle$s opp. = sides)

$O_1 = A + B_1$ (ext. $\angle$ of Δ)

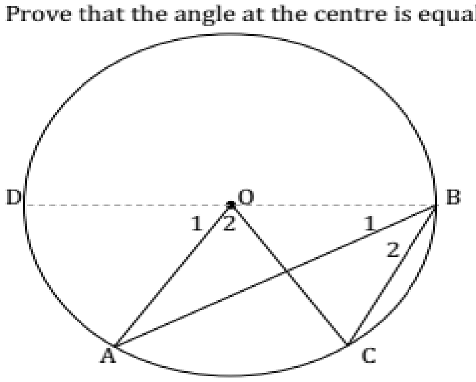
$\therefore O_1 = 2B_1$

Similarly $O_2 = 2B_2$

$O_1 + O_2 = 2B_1 + 2B_2$

$O_1 + O_2 = 2(B_1 + B_2)$

Prove that the angle at the centre is equal to 2 times the angle at the circumference.



Construct line BOD

$OA = OB$ (radii)

$\hat{A} = \hat{B}_1$ ($\angle$s opp. = sides)

$\hat{O}_1 = \hat{A} + \hat{B}_1$ (ext. $\angle$ of Δ)

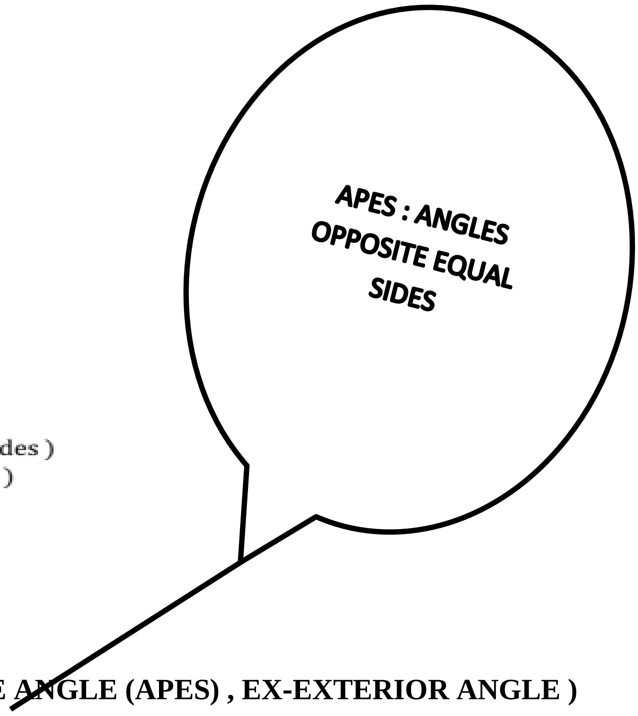
$\therefore \hat{O}_1 = 2\hat{B}_1$

Similarly $\hat{O}_1 + \hat{O}_2 = 2(\hat{B}_1 + \hat{B}_2)$

$\hat{O}_1 + \hat{O}_2 = 2\hat{B}_1 + 2\hat{B}_2$

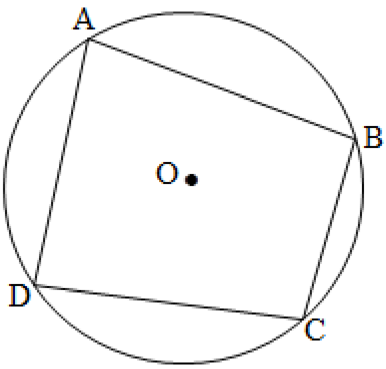
$(\hat{O}_1 + \hat{O}_2) - (\hat{O}_1) = (2\hat{B}_1 + 2\hat{B}_2) - (2\hat{B}_1)$

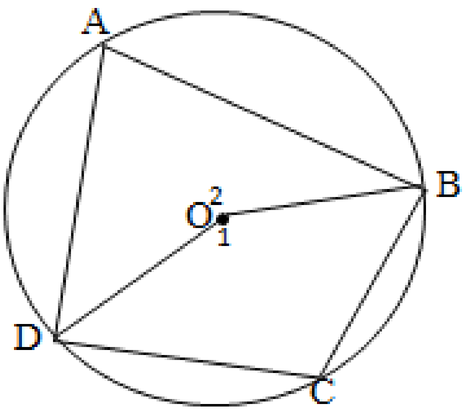
$\hat{O}_2 = 2\hat{B}_2$



HINT :RABASEX (R-RADII ,BAS-BASE ANGLE (APES) , EX-EXTERIOR ANGLE)

3. Prove the theorem that states that the opposite angles of a cyclic quadrilateral are supplementary.





Construction: Draw radii OD and OB

$O_1 = 2\hat{A}$ [< at centre = 2 ×< at circumference]

$O_2 = 2\hat{C}$ [< at centre = 2 × < at circumference]

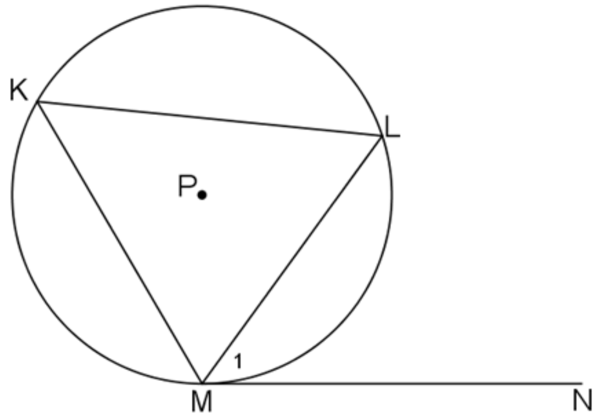
$O_1 + O_2 = 2\hat{A} + 2\hat{C}$

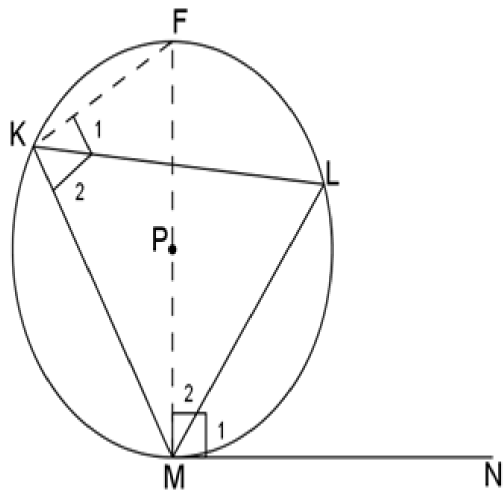
$360^\circ = 2(\hat{A} + \hat{C})$ [revolution]

$180^\circ = \hat{A} + \hat{C}$

HINT : CECERE (CE-CENTRE THEOREM ,CE-CENTRE THEOREM ,RE-REVOLUTION)

4. In the diagram below the circle with centre P, passes through K, L and M. MN is a tangent to the circle at M. KM, KL and ML are joined. Prove the theorem that states that the angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle on the circle subtended by the chord in the opposite segment.





Construction: Draw diameter FM and join KF

$\widehat{K}_1 + \widehat{K}_2 = 90°$ $\qquad$ $(\angle\ in\ a\ semi-circle)$

$\widehat{M}_1 + \widehat{M}_2 = 90°$ $\qquad$ $(Radius \perp tangent)$

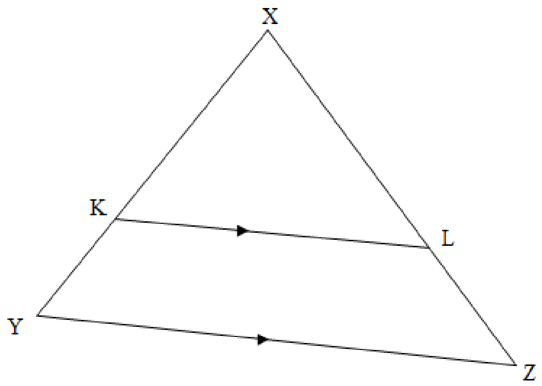
But $\widehat{K}_1 = \widehat{M}_2$ $\qquad$ $(\angle\ in\ same\ circle\ segment)$

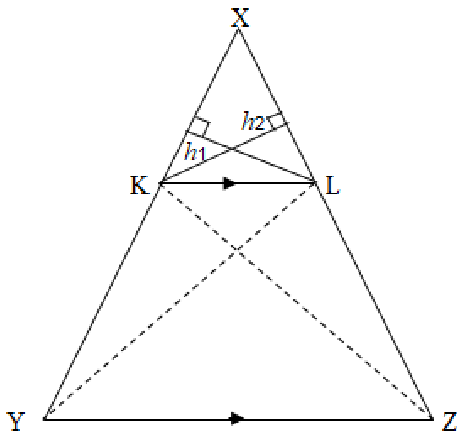
$\therefore \widehat{K}_2 = \widehat{M}_1$

SEMI O RA⊥TA BUTTERFLY (**SEMI**-SEMI CIRCLE,**RATA**-RADIUS PERPENDICULAR TO TANGENT ,**BUTTERFLY** -SAME SEGMENT)

GRADE 12

1. Use the diagram to prove the theorem which states that a line drawn parallel to one side of a triangle divides the other two sides proportionally, that is prove that $\dfrac{XK}{KY} = \dfrac{XL}{LZ}$.





Constr: Join KZ and LY and draw h_1 from K $\perp$ XL and h_2 from L $\perp$ XK:

$$\frac{\text{area } \Delta XKL}{\text{area } \Delta LYK} = \frac{\frac{1}{2} XK \times h_1}{\frac{1}{2} KY \times h_1} = \frac{XK}{KY}$$

$$\frac{\text{area } \Delta XKL}{\text{area } \Delta KLZ} = \frac{\frac{1}{2} XL \times h_2}{\frac{1}{2} LZ \times h_2} = \frac{XL}{LZ}$$

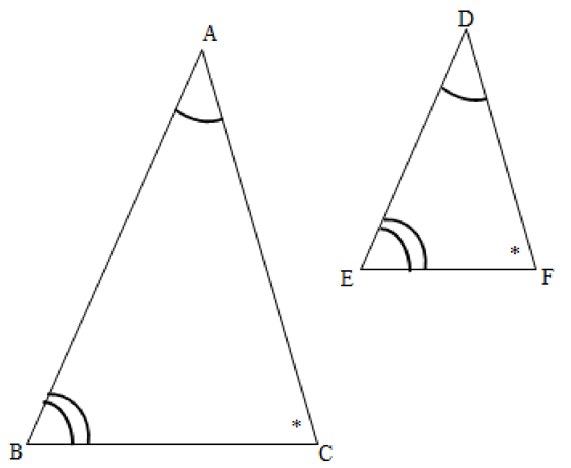
area ΔXKL = area ΔXKL [common]

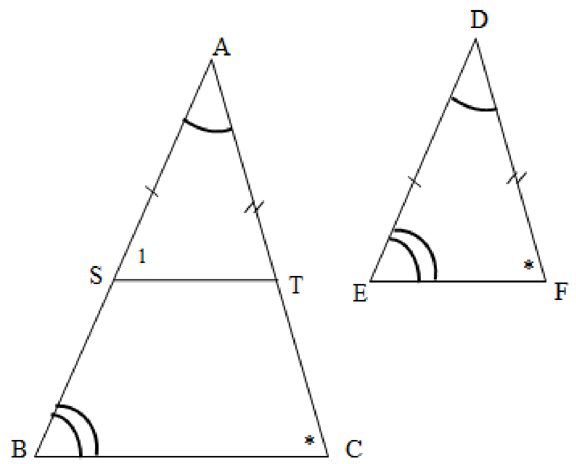
But area ΔLYK = area ΔKLZ [same base & height ; LK || YZ]

$$\therefore \frac{\text{area } \Delta XKL}{\text{area } \Delta LYK} = \frac{\text{area } \Delta XKL}{\text{area } \Delta KLZ}$$

$$\therefore \frac{XK}{KY} = \frac{XL}{LZ}$$

2. In the diagram below $\widehat{A} = \widehat{D}$, $\widehat{B} = \widehat{E}$ and $\widehat{C} = \widehat{F}$. Use the diagrams below to prove the theorem that states $\frac{AB}{DE} = \frac{AC}{DF}$





Construction: Draw ST with S on AB and T on AC such that AS = DE and AT = DF

In ΔAST and ΔDEF

AS = DE	(construction)
$\widehat{A} = \widehat{D}$	(given)
AT = DF	(construction)
$\therefore \Delta AST \equiv \Delta DEF$	(S, ∠, S)
$\therefore \widehat{S}_1 = \widehat{E}$	(from congruency)
$= \widehat{B}$	(given)
∴ ST // BC	(corresp. ∠s =)
$\therefore \frac{AB}{AS} = \frac{AC}{AT}$	(line ll one side of Δ)
$\therefore \frac{AB}{DE} = \frac{AC}{DF}$	(construction)

GUIDELINES, SUMMARY NOTES, & STRATEGIES

WAYS IN WHICH EUCLIDEAN GEOMETRY IS TESTED

1. Completing a statement of a theorem in words.
2. Determining the value of an angle in **two ways:** numerical and / or in terms of the variable(s)
3. **Proofs in riders:** Direct and indirect proofs
4. **Similarity** and **Proportionality** Theorems
   - Proportionality theorem: Question involving parallel lines in proportions, Areas (common angle vs. common vertex/same height)
   - Similarity theorem: AAA, ratios after similar triangles.
5. **Examinable proofs** to be known

1. COMPLETING A STATEMENT OF A THEOREM IN WORDS.

- Know by heart all the theorems and be able to complete the statement.

2. DETERMINING THE VALUE OF AN ANGLE

- Know all the theorems about **lines**, **triangles** and **circles (Centre** group, **non-centre** group, **tangent** group and **cyclic quad** group**)**.
- Every statement must come with a reason and reasons must be stated according to the list of acceptable reasons from the exam guidelines

E.g. base$\angle$'s of an iso. $\triangle$ (unacceptable) the acceptable reason is: $\angle$'s opp = sides

3. PROOFS IN RIDERS

Know how theorems and their converses are being formed in diagrams.

- When given 3 points on the circumference look out for a possibility of a triangle. If one side is produced then you may expect exterior angle of a triangle. If there is a tangent on the circle then there is a possibility of having a Tan Chord Theorem
- When given 4 or 5 points on the circumference then there is a possibility that 4 points may be joined and then there is a cyclic quad. In a case that one side is produced then you may expect exterior angles of a cyclic quad.
- Start with a given angle linking with what is required to prove
- Visualization: Mind picture of diagrams of theorems.

DIRECT AND INDIRECT PROOFS IN RIDERS.

- In Geometry we mostly use angles to prove in questions.

1. **Direct** proof question: Prove A = B

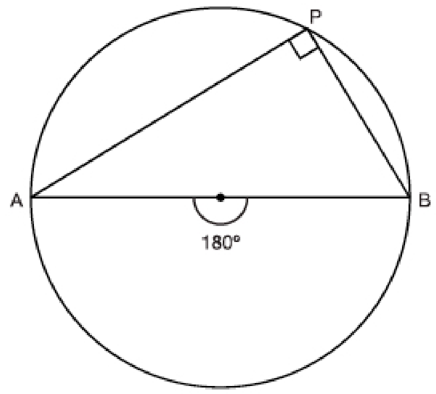
2. **Indirect** proof question: Prove that a line // to another line.

Remember in Euclidean geometry- we mostly use angles to prove. This question is not asking about the angles directly. Here we need to prove sides but using angles i**ndirectly. Why indirectly?** Because we mostly use angles to prove. ∴First, we need to change this question to be direct, and then prove. If we say it must be direct we mean that it must ask to prove angles 1st, then conclude by stating the sides that are parallel

Alternative approach to solving riders

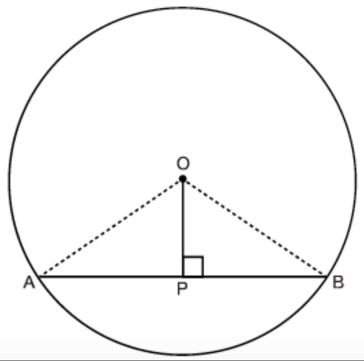
Docto**R** **C**a**P**e **T**own (**DR CPT**)

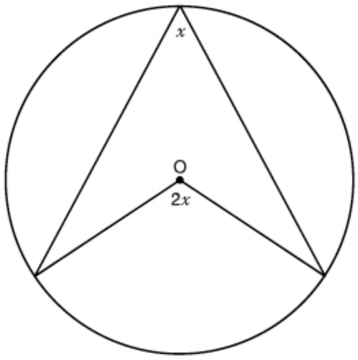
Diameter

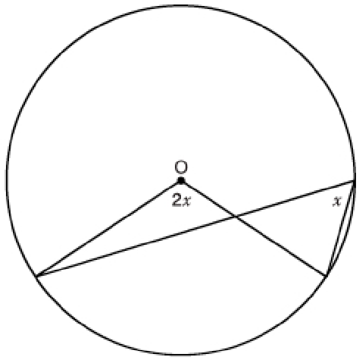


∠'s in semi-circle

Radius



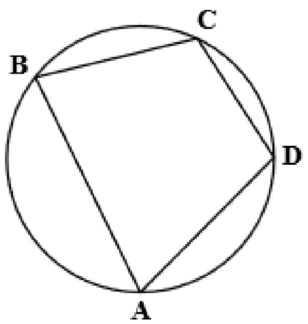




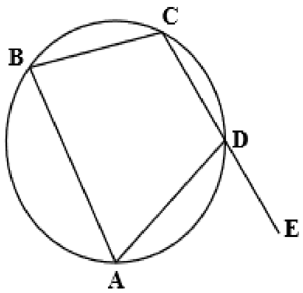
- line from centre to chord
- ∠'s opp equal sides

∠ at centre = 2 ×∠ at circumference

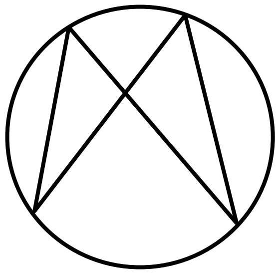
Cyclic quad



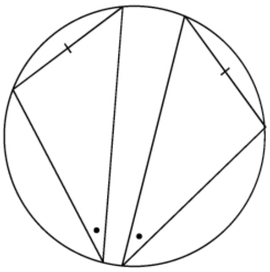
Opp ∠'s of cyclic quad



ext ∠ of cyclic quad

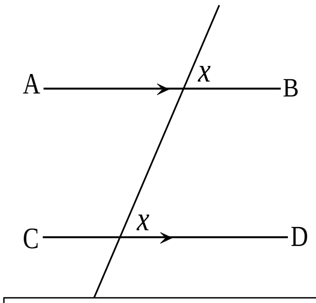


∠'s in the same seg

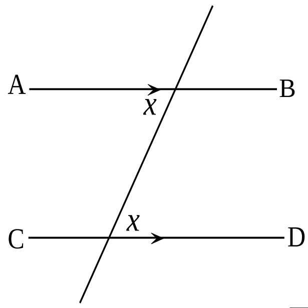


equal chords; equal ∠'s

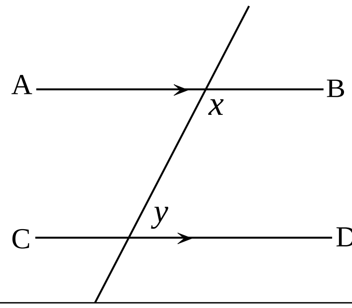
Parallel lines



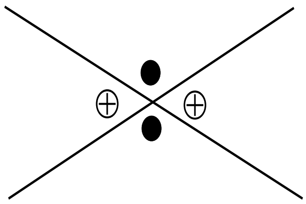
corresp ∠'s ; AB || CD



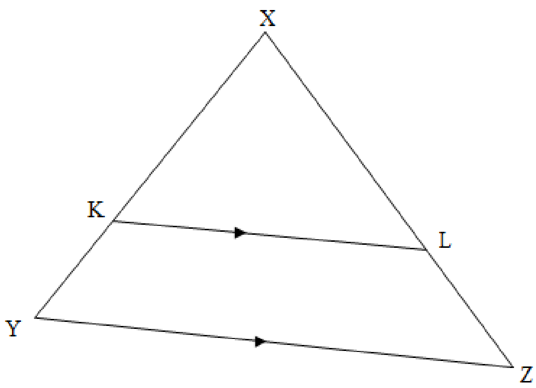
alt ∠'s ; AB || CD



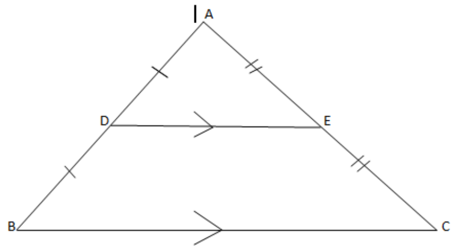
X + Y =180 co-int ∠'s ; AB || CD



vert opp ∠'s =



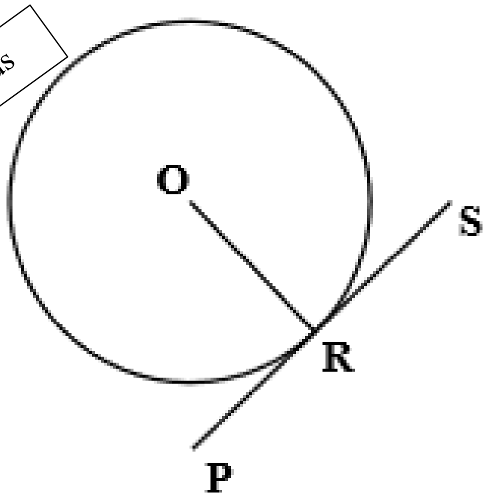
line || one side of Δ

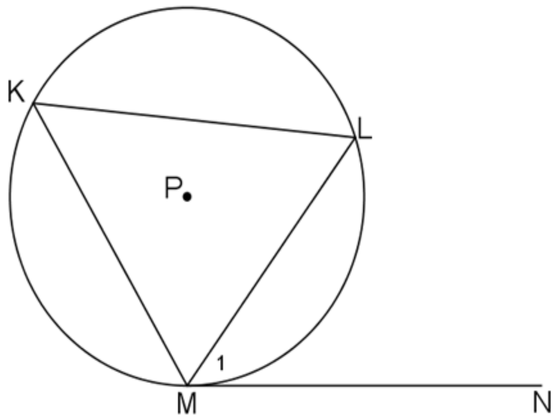


Midpt Theorem, DE||BC

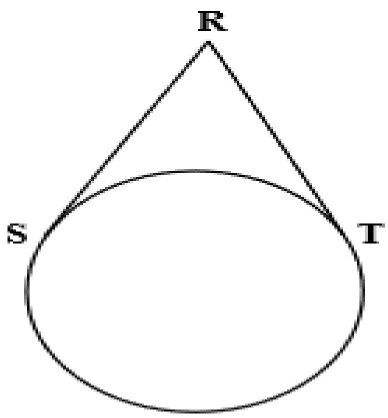
Tangents

tan ⊥radius





tan chord theorem



Tans from common pt

1	**Theorem****	A line drawn parallel to one side of a triangle divides the other two sides proportionally. **(line || one side of Δ OR prop theorem; name || lines)**
	Converse	If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side. **(line divides two sides of Δ in prop)**
	Theorem**	If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar) **(||| Δs OR equiangular Δs)**
	Converse	If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar). **(Sides of Δ in prop)**

Two variables are **proportional** if there is a constant **ratio** between them.

PROPORTIONALITY

Ratio A ratio describes the relationship between two quantities which have the same units. We can use ratios to compare lengths, age, etc. A ratio is a comparison between two quantities of the same kind and has no units.

Example 1: if the length of the base of a triangle is 200 cm and the height is 40 cm, then we can express the ratio between the length of the base and the height of the triangle:

A ratio written as a fraction is usually given in its simplest form.

$$Length\ of\ base : height$$
$$200 : 40$$
$$5 : 1$$

$$\frac{lenght\ of\ base}{height} = \frac{200}{40} = \frac{5}{1}$$

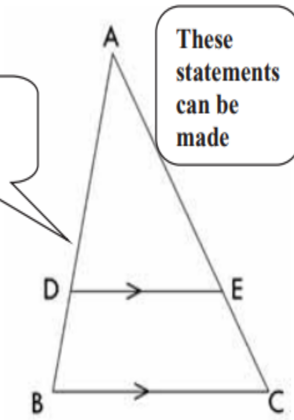
If two or more **ratios** are equal to each other, then we say that they are in the same **proportion**.

Example: If $\frac{AB}{CD} = \frac{5}{10} = \frac{1}{2}$

And $\frac{KL}{MN} = \frac{4}{8} = \frac{1}{2}$

$\therefore \frac{AB}{CD} = \frac{KL}{MN}$

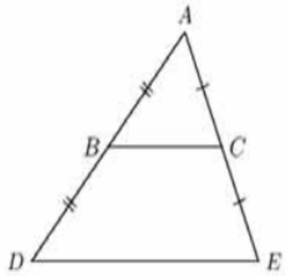
Triangle **Proportionality Theorem**.
If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the line divides these two sides **proportionally**.



Statement	Reason
$\frac{AD}{DB}=\frac{AE}{EC}$	prop theorem DE ∥ BC
$\frac{AB}{DB}=\frac{AC}{EC}$	prop theorem DE ∥ BC
$\frac{AD}{AB}=\frac{AE}{AC}$	prop theorem DE ∥ BC

SPECIAL CASE OF THE CONVERSE PROPORTIONALITY THEOREM: THE MID-POINT THEOREM

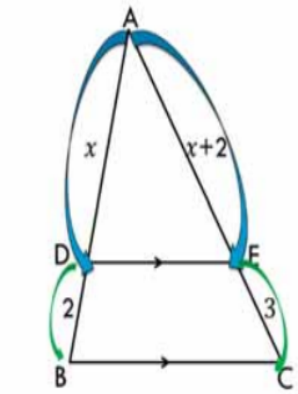
A corollary of the proportion theorem is the mid-point theorem: the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



If AB = BD and AC = CE, then BC∥DE and BC = ½DE.

We also know that $\frac{AC}{CE} = \frac{AB}{BD}$

APPLYING THE PROPRTIONALITY THEOREM:
EXAMPLE 1
In the diagram below, ΔABC has D on AB and E on AC such that DE ∥ BC.
DB = 2 units, EC = 3 units, AD = x units and AE = $x + 2$ units.
Determine the value of x.

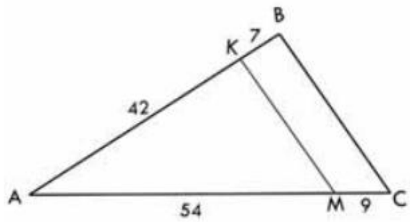


Statement	Reason
$\frac{AD}{DB}=\frac{AE}{EC}$	prop theorem DE ∥ BC
$\frac{x}{2}=\frac{x+2}{3}$	
$2(x+2) = 3x$	
$2x+4 = 3x$	
$4 = x$	

CONVERSE OF THE PROPORTIONALITY THEOREM:

EXAMPLE 2

In the diagram : KB = 7 units; AK = 42 units; AM = 54units and MC = 9 units.
Prove that KM is parallel to BC.



We need to prove that KM divide the sides of the Δ ABC proportionally (*in other words*: $\frac{AK}{KB} = \frac{AM}{MC}$) :

Let's investigate:

$$\frac{AK}{KB} = \frac{42}{7} = 6$$

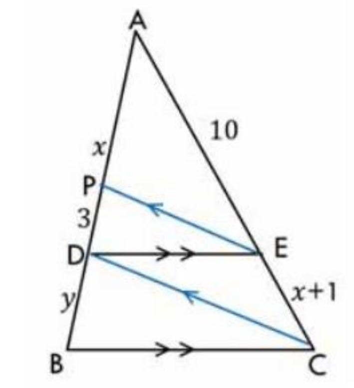
$$\frac{AM}{MC} = \frac{54}{9} = 6$$

This could come in handy if you want to prove TWO lines are parallel!

$\therefore$ KM $\parallel$ BC

EXAMPLE 3

In the diagram, ΔABC has D and P on AB and E on AC such that DE $\parallel$ BC and PE $\parallel$ DC
DB = y units, DP = 3 units, AP = x units, AE = 10 units and AE = $x + 1$ units.
Determine the value of x.



$\frac{AP}{DP} = \frac{AE}{EC}$ prop theorem PE||DC

$\frac{x}{3} = \frac{10}{x+1}$

$x(x+1) = 30$

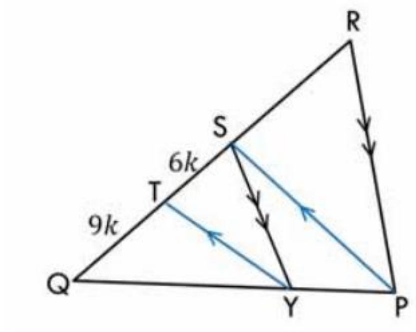
$x^2 + x - 30 = 0$

$(x+6)(x-5) = 0$

$x \neq -6 \ or \ x = 5$

EXAMPLE 4

In the diagram below, ΔPQR has T and S on RQ and Y on QP such that TY|| SP and SY|| PR
If $\frac{QT}{TS} = \frac{9}{6}$; determine the ratio of $\frac{TS}{SR}$

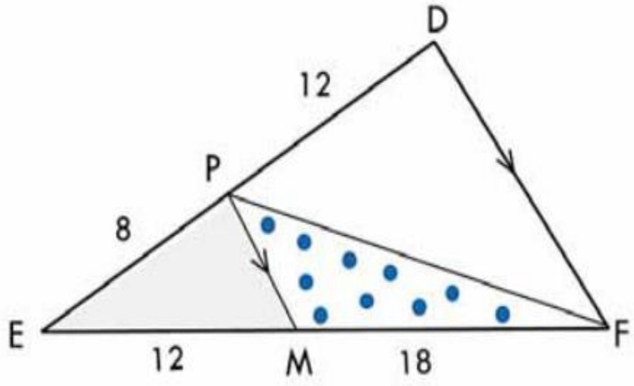


Statement	**Reason**
$\frac{QY}{YP} = \frac{QT}{TS}$	prop theorem TY|| SP
$\frac{QX}{XP} = \frac{\mathbf{9k}}{\mathbf{6k}} = \frac{\mathbf{3}}{\mathbf{2}}$	
$\frac{QY}{YP} = \frac{QS}{SR}$	prop theorem SY|| PR
$\frac{3}{2} = \frac{9k+6k}{SR}$	
$3SR = 30k$	
$SR = 10k$	
$\frac{TS}{SR} = \frac{6k}{10k} = \frac{6}{10}$	

AREA OF TRIANGLES IN PROPORTIONALITY

EXAMPLE 5

In the diagram is ΔEFD with PM parallel to DF.
PD=12 units, EP = 8 units, EM =12 units and MF=18units



5.1 Determine the ratio of: $\frac{area\,\Delta PEM}{area\,\Delta PMF}$ 5.2 Determine the ratio of: $\frac{area\,\Delta PEM}{area\,\Delta DEF}$

- There are TWO known formulas for the area of a Δ.
- We have to decide which formula works best in a given question.

1) Area of Δ= $\frac{1}{2}\times base \times height$ → *use when two Δs have a common height.*
2) Area of Δ= $\frac{1}{2}\times ab\ sinC$ → *use when two Δs have a common angle.*

5.1

$$\frac{area\,\Delta PEM}{area\,\Delta PMF}=\frac{\frac{1}{2}\times EM\times h_P}{\frac{1}{2}\times MF\times h_P}\text{.... common height}$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF}=\frac{EM}{MF}$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF}=\frac{12}{18}$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF}=\frac{2}{3}$$

Always simplify

5.2

$$\frac{area\,\Delta PEM}{area\,\Delta DEF}=\frac{\frac{1}{2}\times EM\times PE\times sinE}{\frac{1}{2}\times EF\times ED\times sinE}\text{... common angle}$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF}=\frac{EM\times PE}{EF\times ED}$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF}=\frac{12\times 8}{30\times 20}$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF}=\frac{96}{600}$$

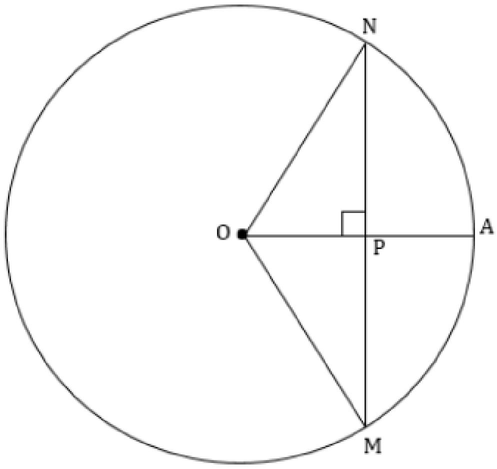
$$\frac{area\,\Delta PEM}{area\,\Delta DEF}=\frac{4}{25}$$

GRADE 12

LEVEL 1 AND LEVEL 2 QUESTIONS

QUESTION 1

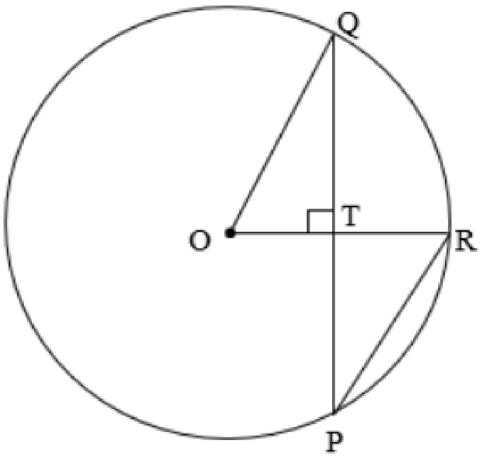
1. In the diagram, O is the centre of the circle NAM and OPA ⊥ MPN. MN = 48 units and OP = 7 units.



1.1 Calculate with reasons the length of PA. (5)

QUESTION 2

In the diagram below, PQ is the chord of circle O. OR ⊥ PQ and OR intersect PQ at T. If the radius of the circle is 13 cm and PT = 12 cm.



Calculate the length of:

2.1 PQ (2)

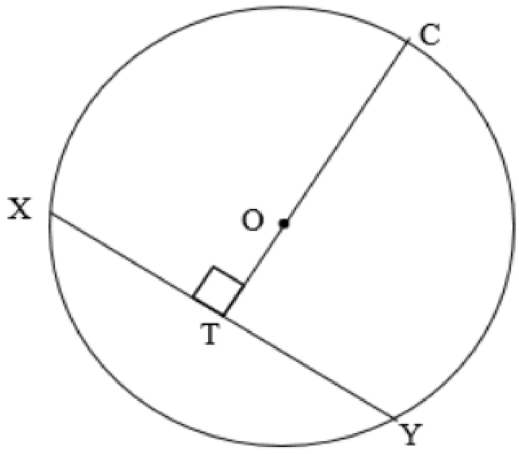
2.2 PR (4)

[6]

QUESTION 3

In the diagram drawn below, O is the centre of the circle XCY. COT ⊥ XY.

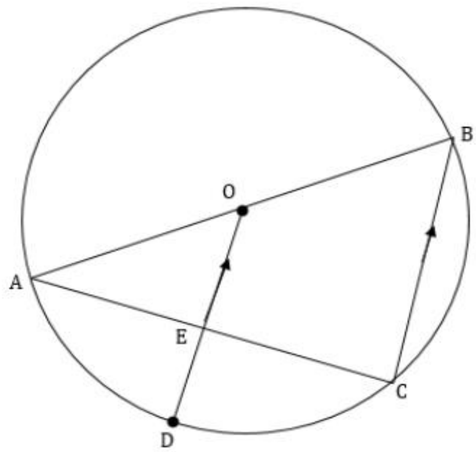
OC = r and XY = $\frac{3}{2}r$



Prove, stating reasons, that CT $= \frac{4+\sqrt{7}}{4}r$ (6)

QUESTION 4

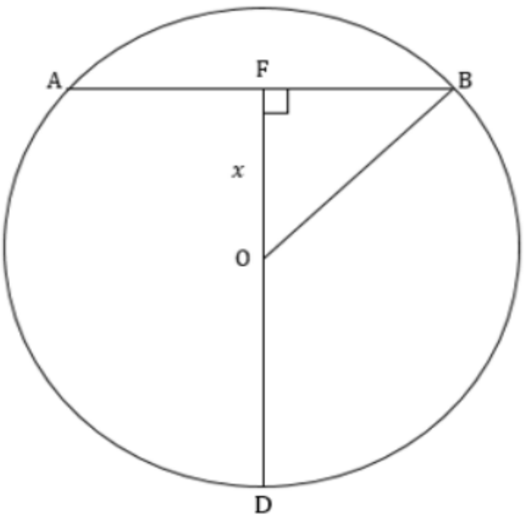
AB is a diameter of then circle ABCD. OD is draw parallel to BC and meets AC at E.



If the radius is 10 cm and AC = 16 cm, calculate the length of ED. (5)

QUESTION 5

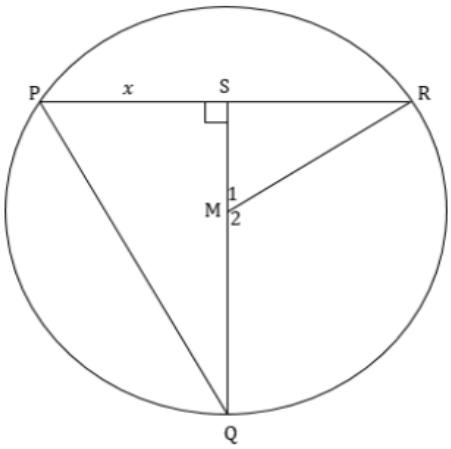
In the diagram, O is the centre of the circle ABD. F is a point on chord AB such that DOF⊥ AB. AB = FD = 8 cm and OF = x cm.



Determine the length of the radius of the circle. (5)

QUESTION 6

In the diagram, PR and PQ are equal chords of the circle with centre M. QS is perpendicular to PR at S. PS = x cm and MR is drawn.



6.1 Express, giving reasons, QS in terms of x. (5)

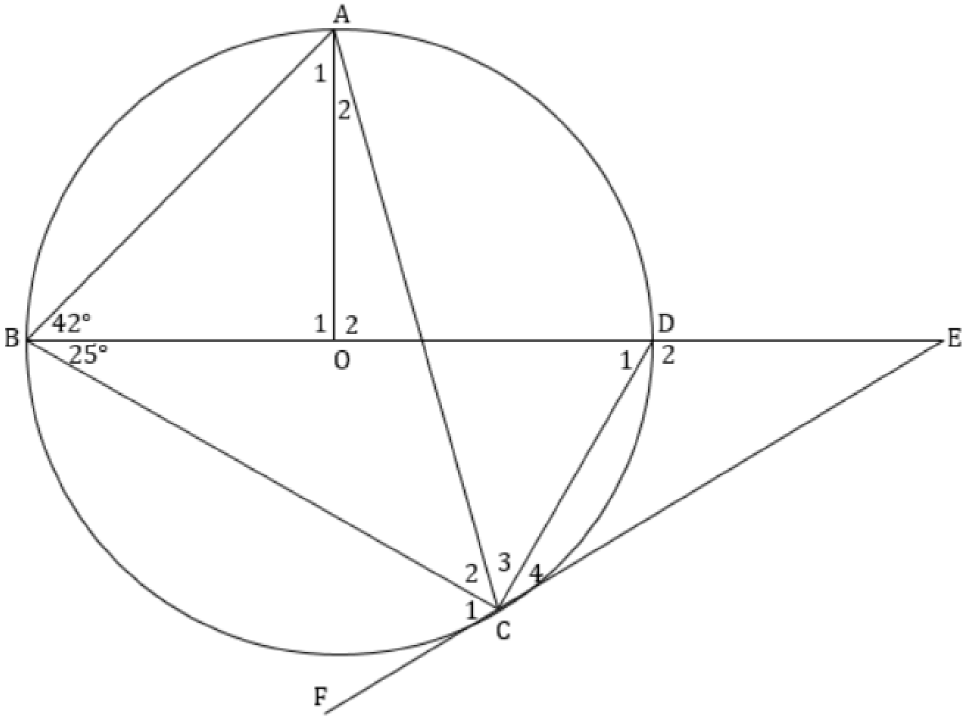
6.2 If $x = \sqrt{12}$ and MS = 1 unit, calculate the length of the radius of the circle. (2)

6.3 Calculate, giving reasons the size of $\widehat{P}$ (5)

[12]

QUESTION 7

In the diagram below, the circle with centre O passes through A, B, C and D such that BOD is a diameter. BD is extended to E such that FCE is a tangent to the circle at C. $A\hat{B}E = 42°$ and $D\hat{B}C = 25°$

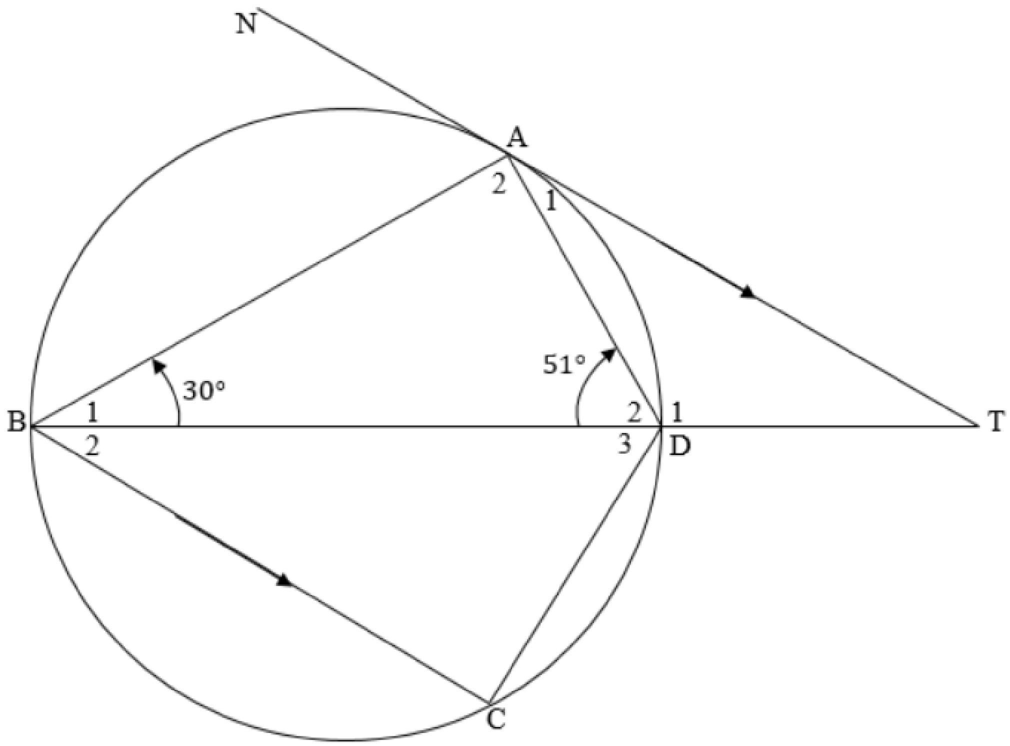


Calculate:

7.1	$B\hat{C}D$	(2)
7.2	$\hat{A}_1$	(2)
7.3	$\hat{O}_2$	(2)
7.4	$\hat{C}_4$	(2)
		[8]

QUESTION 8

In the diagram, TAN is a tangent to the circle at A. ABCD is a cyclic quadrilateral. BD is drawn and produced to meet the tangent at T. $\hat{B}_1 = 30°$ and $\hat{D}_2 = 51°$, TAN ∥ CB.



Calculate giving reasons, the size of:

8.1 $\hat{A}_1$ (2)

8.2 $\hat{T}$ (2)

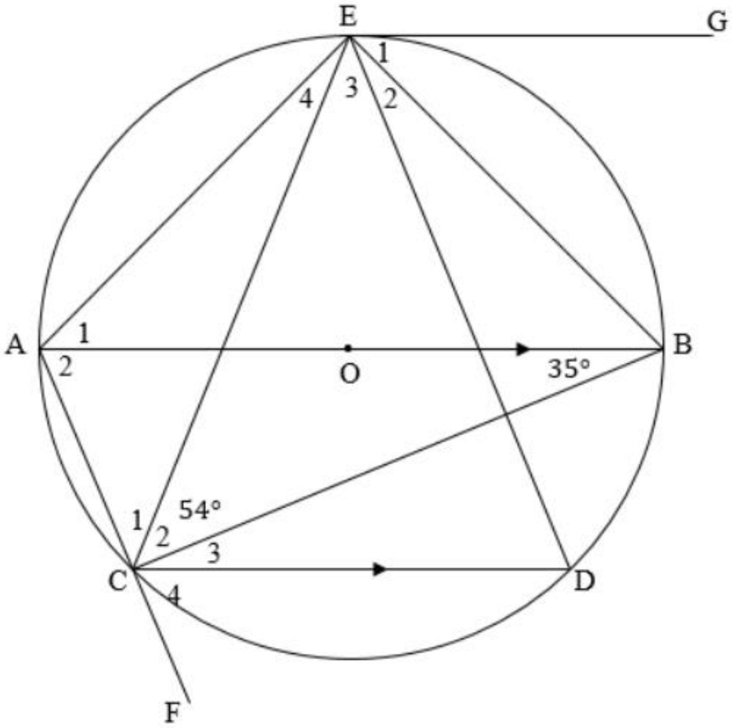
8.3 $\hat{B}_1$ (2)

8.4 $\hat{C}$ (2)

[8]

QUESTION 9

O is the centre of the circle in the diagram with chord CD parallel to diameter AB. AC is produced to F and EG is a tangent to the circle. $A\hat{B}C = 35°$ and $\hat{C}_2 = 54°$.



Calculate, with reasons, the sizes of the following angles:

9.1 $\hat{E}_1$ (2)

9.2 $\hat{C}_2$ (2)

9.3 $\hat{C}_3$ (2)

9.4 $A\hat{E}D$ (2)

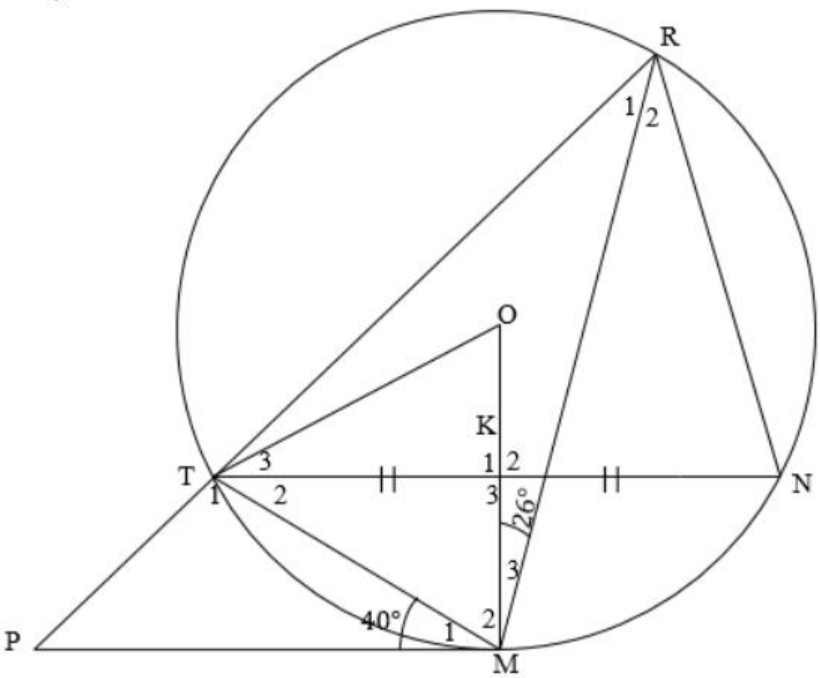
3.5 $\hat{E}_3$ (3)

[11]

QUESTION 10

In the diagram below, O is the centre of circle TRNM. MP is a tangent to the circle at M such that RT produced meet at P. OM intersects TN at K. K is the midpoint of TN.

$P\hat{M}T = 40°$ and $\hat{M}_3 = 26°$



Calculate, with reasons, the size of:

10.1 $T\hat{O}M$ (4)

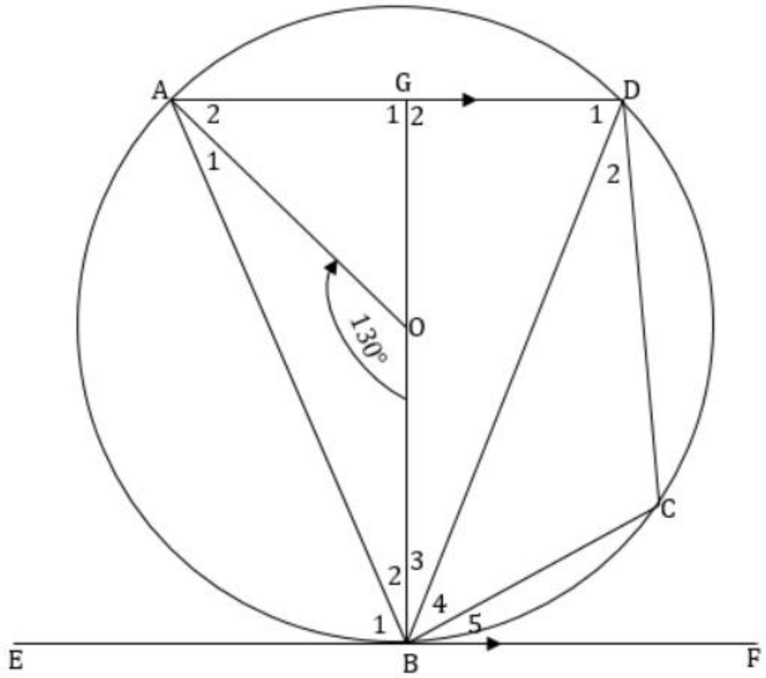
10.2 $\hat{N}$ (4)

10.3 $\hat{T}_3$ (3)

[11]

QUESTION 11

In the diagram below, the circle having centre O, passes through A, B, C and D, with $A\hat{O}B = 130°$. EBF is a tangent to the circle at B with EF∥AD. BOG is a straight line.



Calculate, with reasons, the size of:

11.1 $\hat{D}_1$ (2)

11.2 $\hat{B}_1$ (2)

11.3 $B\hat{A}D$ (1)

11.4 $\hat{C}$ (2)

11.5 $\hat{B}_3$ (3)

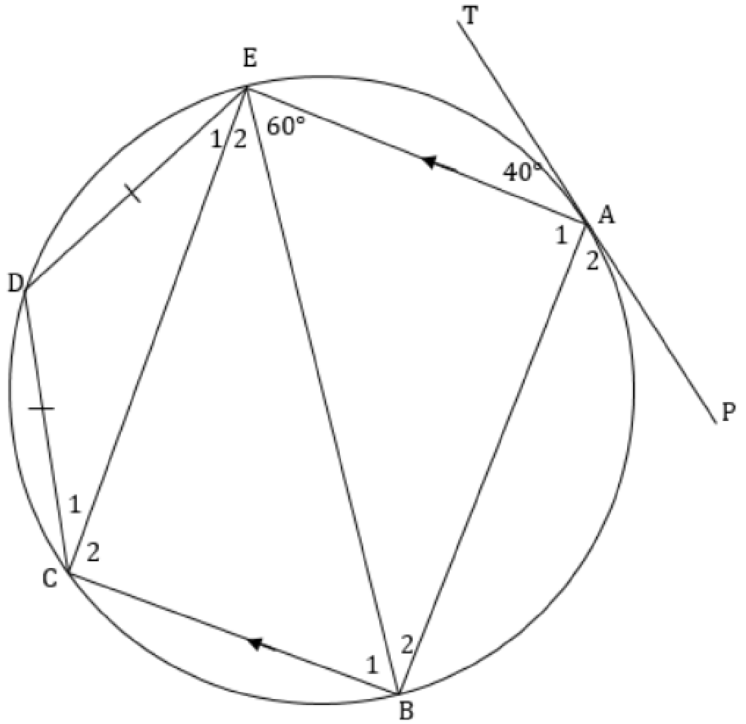
11.6 Calculate the length of GD, if $AD = \frac{\sqrt{7}}{2}$ units (3)

[13]

QUESTION 12

In the diagram below, TAP is a tangent to circle ABCDE at A. AE∥BC and DC = DE.

$T\hat{A}E = 40°$ and $A\hat{E}B = 60°$



12.1 Identify TWO cyclic quadrilaterals. (2)

12.2 Calculate, with reasons, the size of:

12.2.1 $\hat{B}_2$ (2)

12.2.2 $\hat{B}_1$ (2)

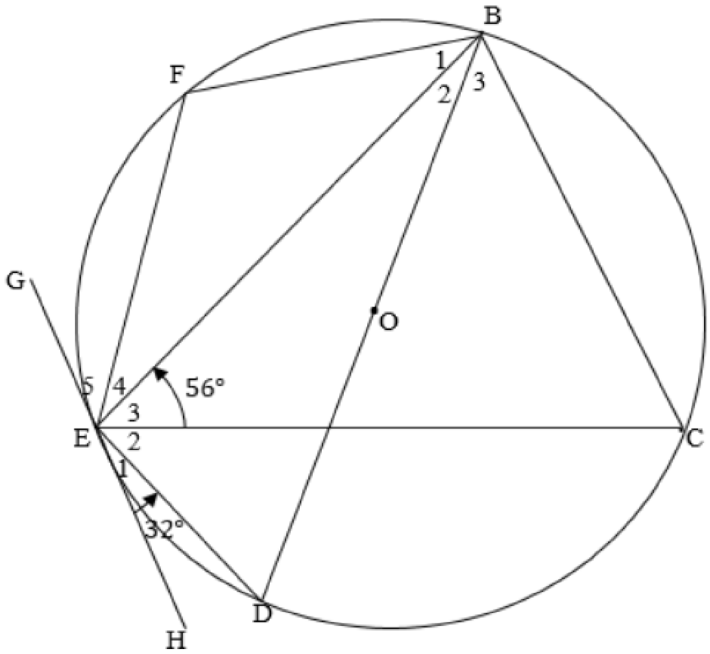
12.2.3 $\hat{D}$ (2)

12.3.4 $\hat{E}_1$ (3)

[11]

QUESTION 13

In the diagram below, O is the centre of the circle. BD is a diameter of the circle. GEH is a tangent to the circle at E. F and C are two points on the circle, then FB, FE, BC, CE and BE are drawn. $\hat{E}_1 = 32°$ and $\hat{E}_3 = 56°$.



Calculate, with reasons, the size of:

13.1 $\hat{E}_2$ (2)

13.2 $E\hat{B}C$ (3)

13.3 $\hat{F}$ (4)

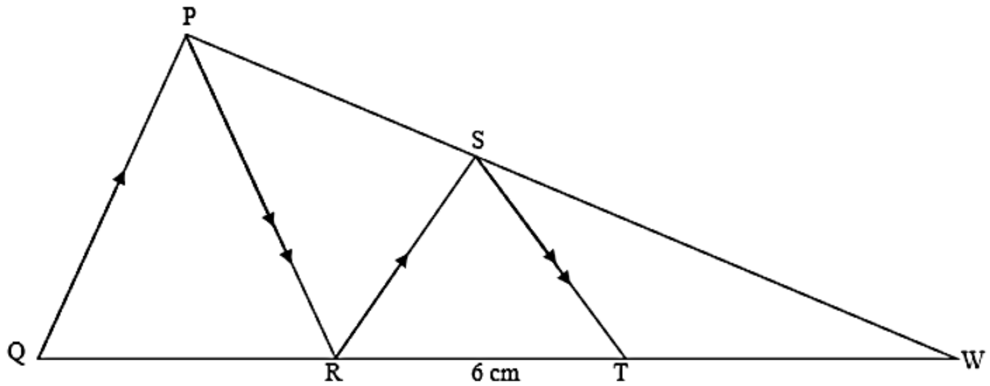
[9]

GRADE 12

LEVEL 3 AND LEVEL 4 QUESTIONS

QUESTION 1

In ΔPQW, S is a point on PW and R is a point on QW such that $SR \parallel PQ$. T is a point on QW such that $ST \parallel PR$. $RT = 6$ cm, $WS : SP = 3 : 2$.



Calculate:

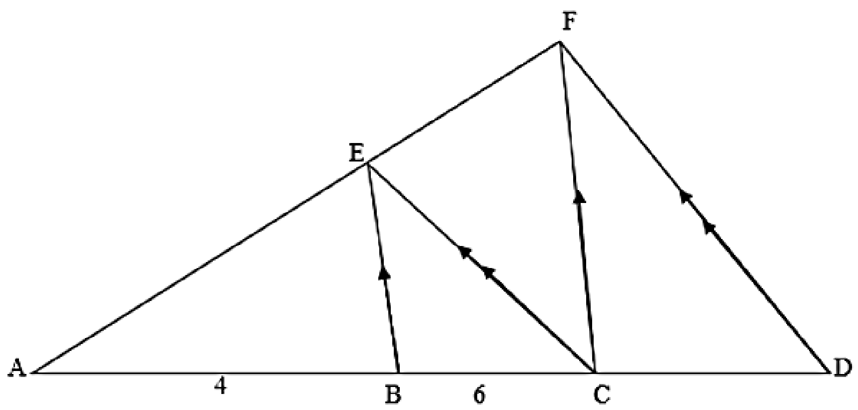
1.1 WT (3)

1.2 WQ (4)

[7]

QUESTION 2

In ΔADF, $DF \parallel CE$ and $CF \parallel BE$. If $AB = 4$ units and $BC = 6$ units.



Calculate:

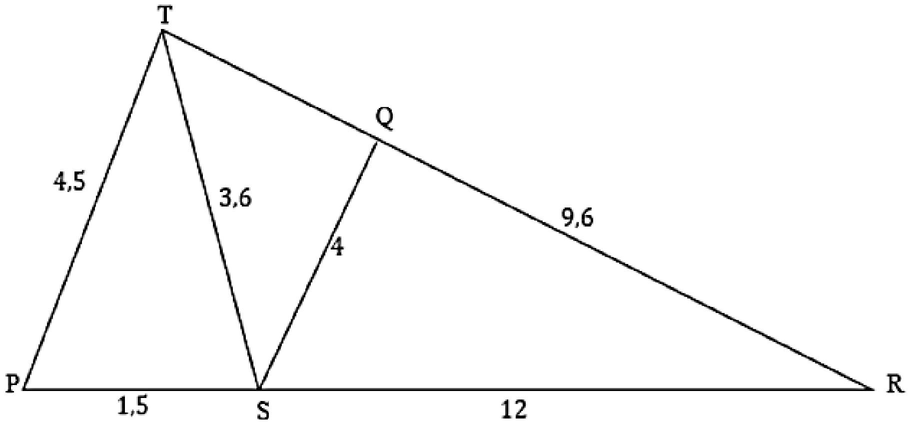
2.1 The length of CD. (3)

2.2 The numerical value of: $\frac{\text{Area of } \Delta FEC}{\text{Area of } \Delta FAD}$ (4)

[7]

QUESTION 3

In the diagram, TRP is a straight with TP = 4,5 units, Q and S are points on TR and PR respectively. QR = 9,6 units, QS = 4 units, TS = 3,6 units, PS = 1,5 units and SR = 12 units.



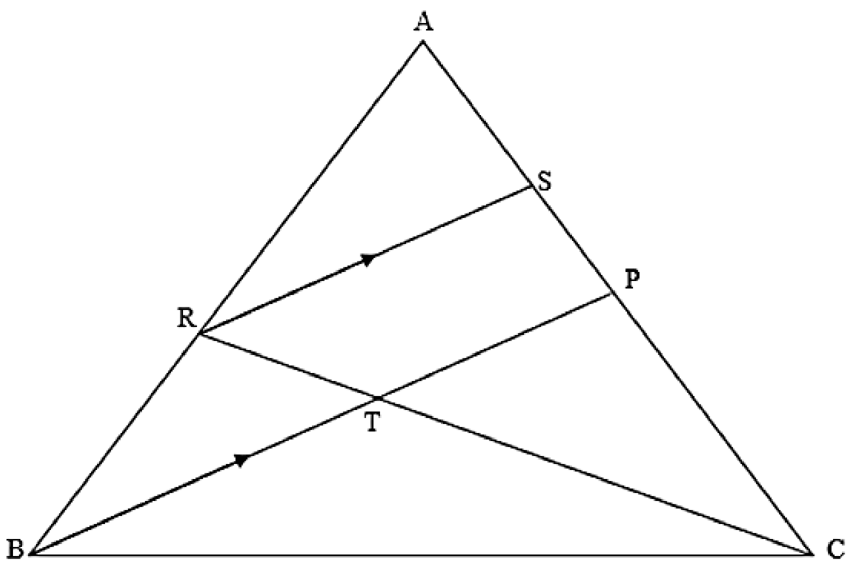
3.1 Prove that PT is a tangent to the circle which passes through the points T, S and R. (7)

3.2 Calculate the length of TQ. (5)

[12]

QUESTION 4

In the diagram below, P is the midpoint of AC in ΔABC. R is a point on AB such that RS||BP and $\frac{AR}{AB} = \frac{3}{5}$. RC cuts at T.



Determine, giving reasons, the following ratios:

4.1 $\frac{AS}{SC}$ (4)

4.2 $\frac{RT}{TC}$ (3)

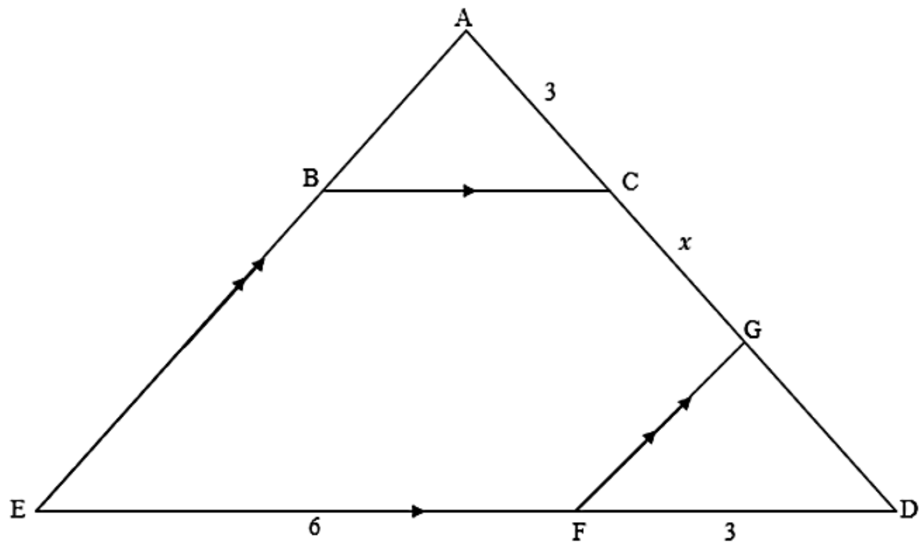
4.3 $\frac{\text{Area of }\Delta RSA}{\text{Area of }\Delta RSC}$ (2)

4.4 $\frac{\text{Area of }\Delta TPC}{\text{Area of }\Delta RSC}$ (4)

[11]

QUESTION 5

In the diagram, ADE is a triangle having BC||ED and AE||GF. It is also given $AB: BE = 1: 3$, $AC = 3$ units, $EF = 6$ units, $FD = 3$ units and $CG = x$ units.



Calculate, giving reasons:

5.1 The length of CD. (3)

5.2 The value of x. (4)

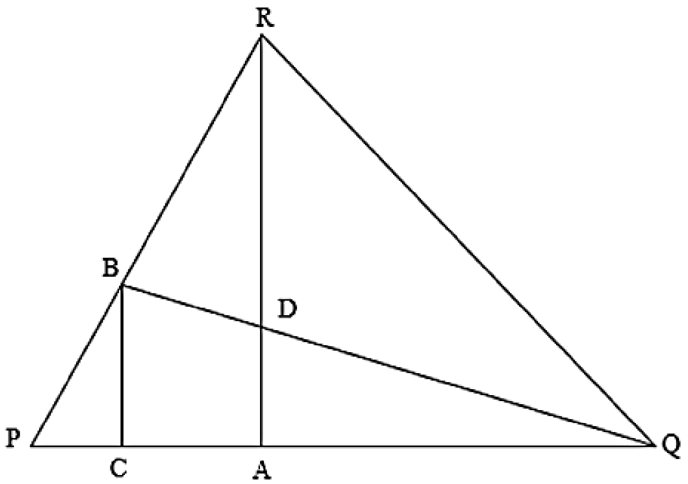
5.3 The length of BC. (5)

5.4 The value: $\frac{\text{Area of }\Delta ABC}{\text{Area of }\Delta GFD}$ (5)

[17]

QUESTION 6

$\frac{PA}{PQ} = \frac{4}{9}$ and $\frac{PB}{BR} = \frac{1}{2}$. BC||RA.



Determine:

6.1 BD: DQ (5)

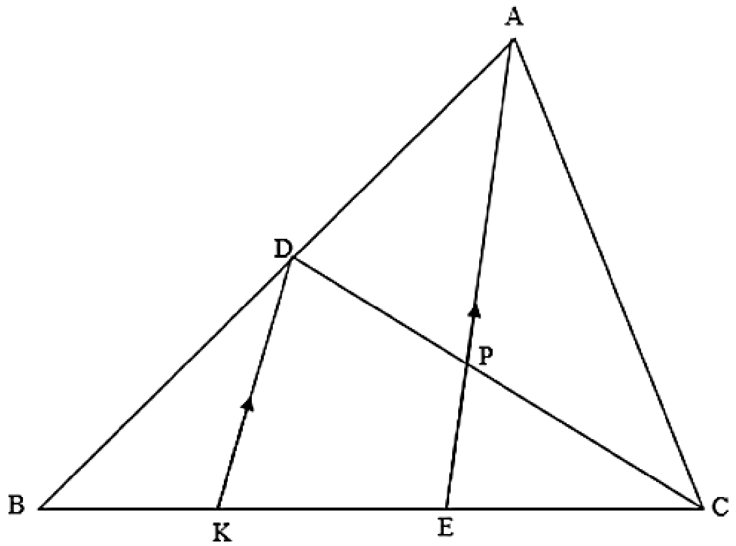
6.2 $\frac{\text{Area of } \Delta PRA}{\text{Area of } \Delta QRA}$ (3)

6.3 $\frac{\text{Area of } \Delta BQC}{\text{Area of } \Delta RPQ}$ (6)

[14]

QUESTION 7

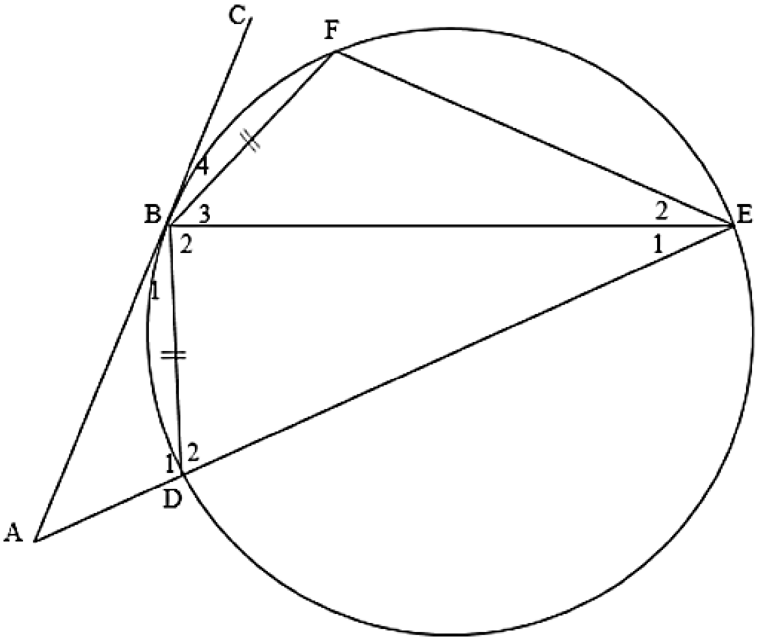
D and E are points on sides AB and BC respectively of ΔABC such that AD: DB = 2: 3 and BE = $\frac{5}{3}$EC. If DK||AE and AE and CD intersect at P, find the ratio of CP: PD.



[5]

QUESTION 8

In the diagram, ABC is a tangent to the circle at B. BDEF is a cyclic quadrilateral with DB = BF. BE is drawn and ED produced meets the tangent at A.



Prove that:

8.1 $\hat{B}_1 = \hat{E}_2$ (3)

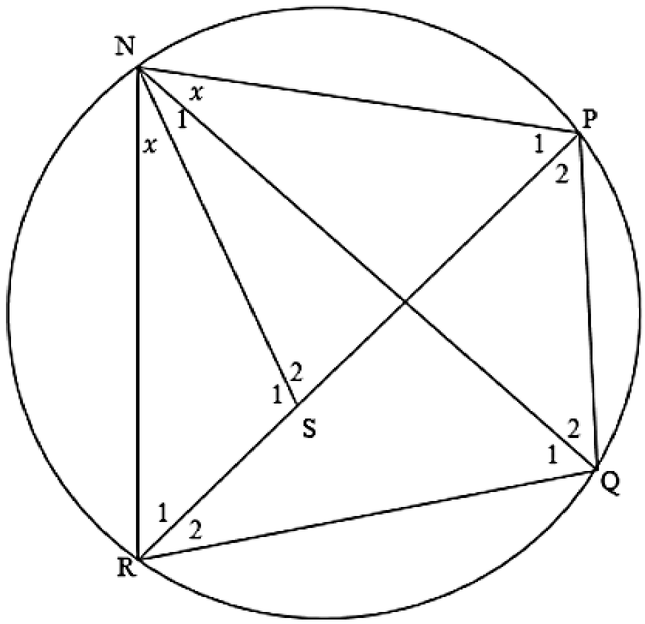
8.2 $\Delta BDA ||| \Delta EFB$ (4)

8.3 $BD^2 = AD.EF$ (2)

[9]

QUESTION 9

In the diagram below, NPQR is a cyclic quadrilateral with S, a point on PR. N and S are joined and $R\hat{N}S = P\hat{N}Q = x$.



Prove that:

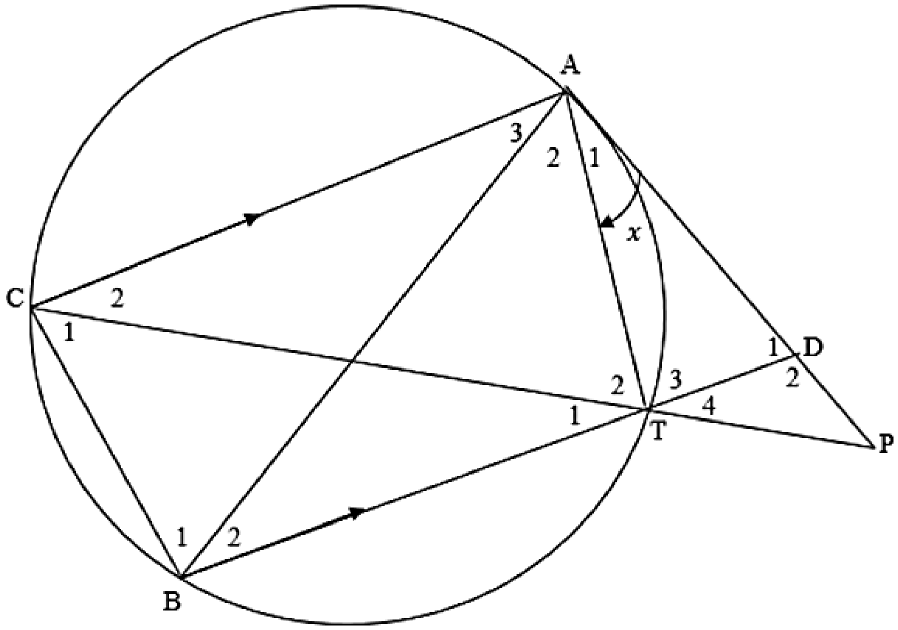
9.1 $\Delta NSR ||| \Delta NPQ$ (3)

9.2 $\Delta NQR ||| \Delta NPS$ (3)

[5]

QUESTION 10

In the diagram below, DA is a tangent to the circle ACBT at A. CT and AD are produced to meet at P. BT is produced to cut PA at D. AC, CB, AB and AT are joined. AC||BD. Let $\hat{A}_1 = x$.

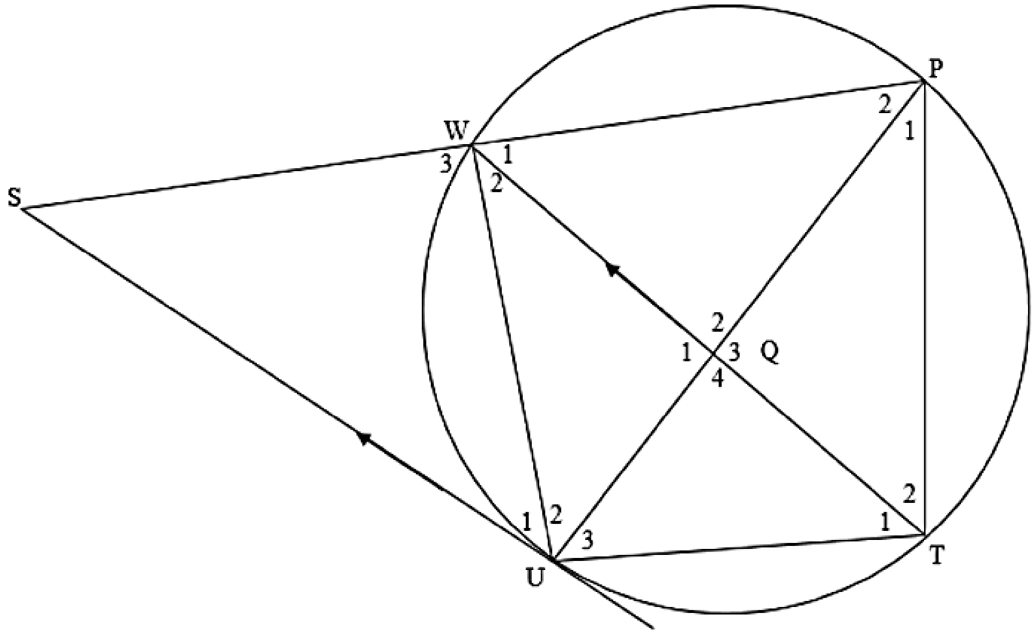


10.1	Prove that $\Delta ABC ||| \Delta ADT$.	(6)
10.2	Prove that PT is a tangent to the circle ADT at T.	(3)
10.3	Prove that $\Delta ATP ||| \Delta TDP$.	(3)
10.4	If $AD = \frac{2}{3} AP$, show that $AP^2 = 3PT^2$.	(4)

[16]

QUESTION 11

In the diagram below, PWUT is a cyclic quadrilateral with WU = TU. Chord WT and PU intersect at Q. PW is extended to S such that US||TW.



Prove that:

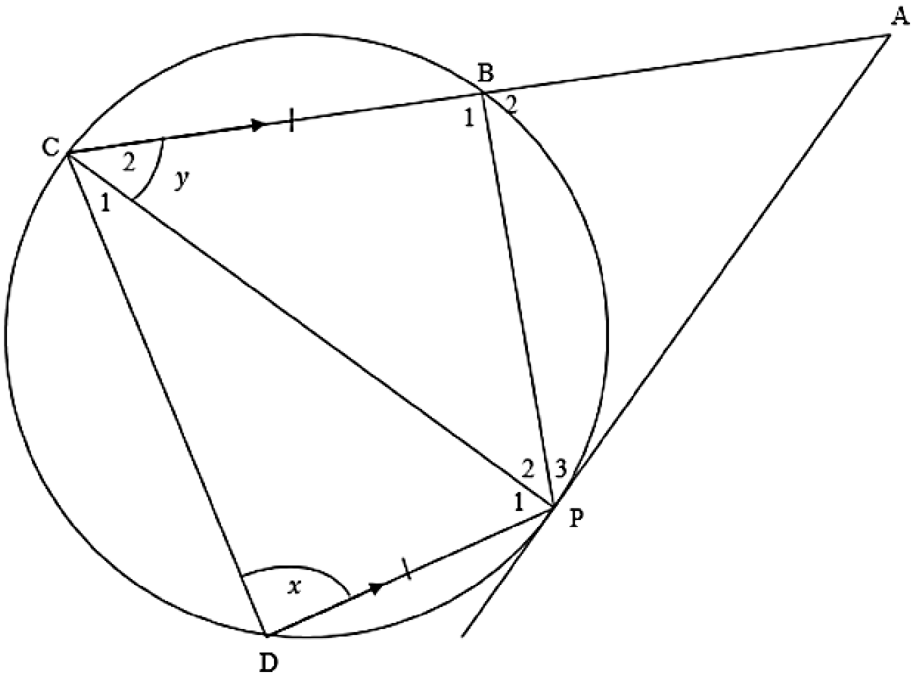
11.1 US is a tangent to the circle PWUT at U. (5)

11.2 ΔSPU|||ΔSUW (4)

[9]

QUESTION 12

AP is a tangent to the circle at P. CP||DP and CB $=$ DP. CBA is a straight line. Let $\hat{D} = x$ and $\hat{C}_2 = y$.



Prove, with reasons that:

12.1 $\Delta APC|||\Delta ABP$ (4)

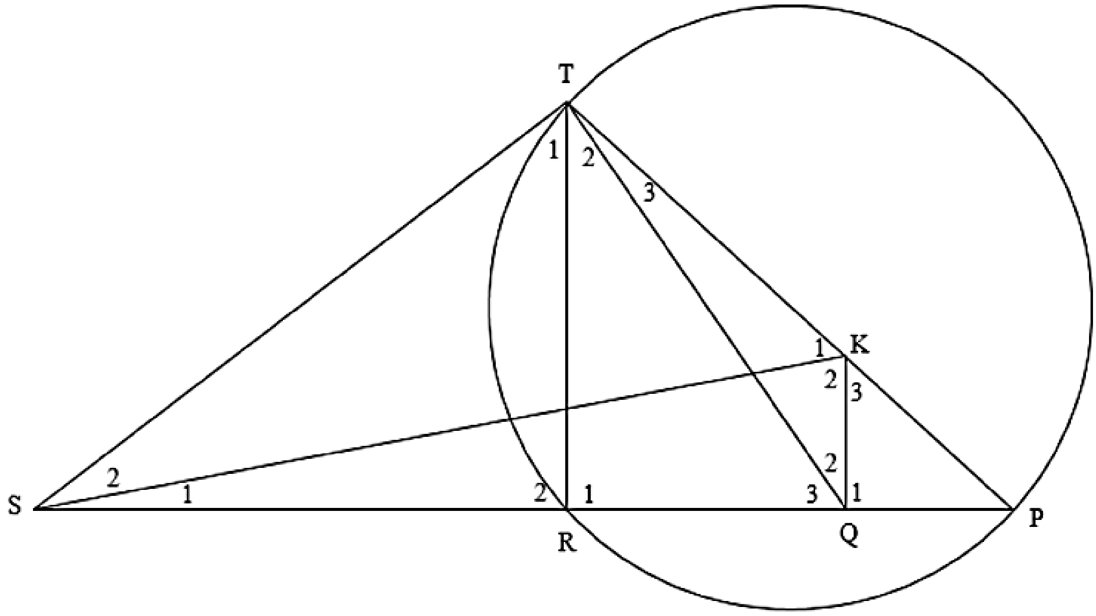
12.2 $AP^2 = AB \times AC$ (1)

12.3 $\Delta APC|||\Delta CDP$ (4)

[9]

QUESTION 13

In the diagram below, ST is a tangent to circle TRP. PT is a diameter, SRQP is a secant. K is a point on PT such that $PK:KT = 1:2$ and $PR = \sqrt{18}$ units and $PQ = \sqrt{2}$ units.



13.1 Prove that:

13.1.1 RT||QK (4)

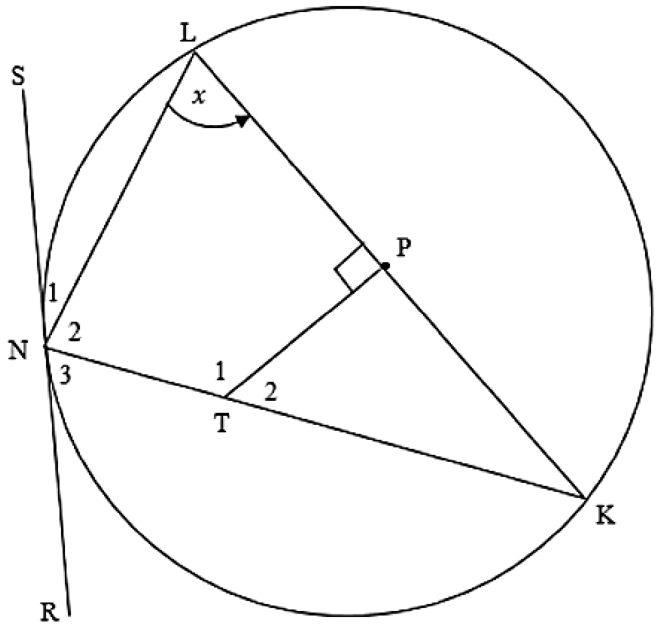
13.1.2 TKQS is a cyclic quadrilateral. (5)

13.1.3 ΔQRT|||ΔKTS (4)

[13]

QUESTION 14

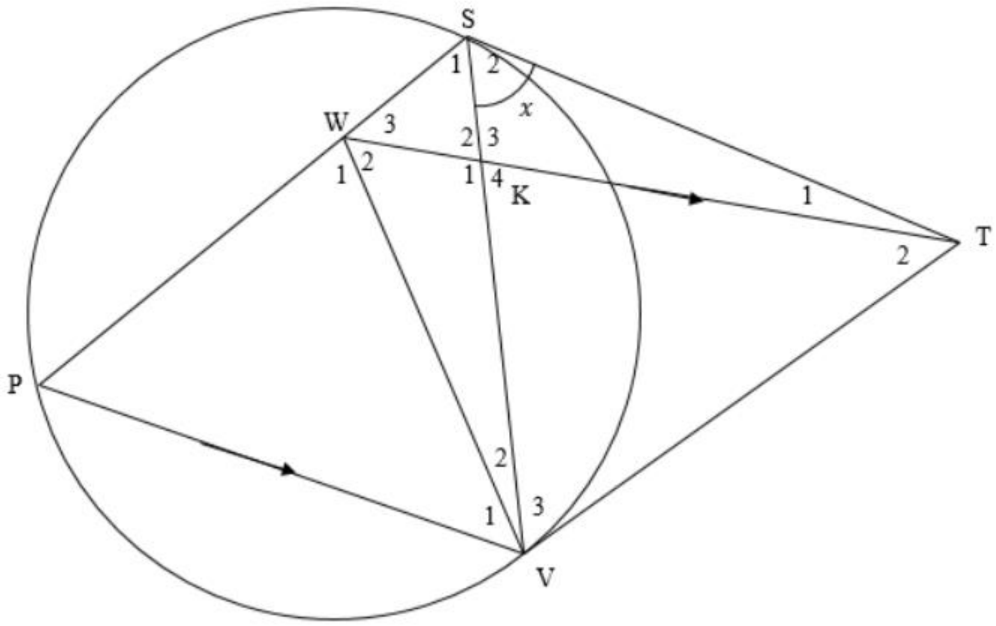
In the diagram, LK is a diameter of the circle with centre P. RNS is a tangent to the circle at N. T is a point on NK and TP ⊥ KL. $P\hat{L}N = x$.



14.1	Prove that TPLN is a cyclic quadrilateral.	(3)
14.2	Determine, giving reasons, the size on $\hat{N}_1$ in terms of x.	(3)
14.3	Prove that:	
	$\Delta KTP \mid\mid\mid \Delta KLN$	(3)
		[9]

QUESTION 15

In the diagram, ST and VT are tangents to the circle at S and V respectively. P is a point on the circle and W is a point on chord PS such that WT is parallel to PV. SV and WV are drawn. WT intersects SV at K. Let $\hat{S}_2 = x$.



15.1 Write down, with reasons, THREE other angles each equal to x. (6)

15.2 Prove, with reasons, that:

15.2.1 WSTV is a cyclic quadrilateral. (2)

15.2.2 ΔWPV is isosceles (4)

15.2.3 ΔWPV $|||$ ΔTSV (3)

[15]